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REPORT No. 880

An Analytical Treatment of The Orientation of A Photogrammetric Camera

HELLMUT H. SCHMID

DEPARTMENT OF THE ARMY PROJECT No. 503-06-011
ORDNANCE RESEARCH AND DEVELOPMENT PROJECT No. TB3-0538C

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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ABSTRACT

The advances made in the development of high speed electronic computing and equipment have increased the importance of analytical methods in precision photogrammetry.

The analytical treatment of orientation and calibration of a photogrammetric camera is discussed from the point of view of the geometrical configuration as well as the least squares adjustment. The solution can be applied to measuring methods in both ground and aerial photogrammetry. The projective relation between two planes is interpreted as a special case of the general problem, which deals with the relation between the plane of the plate and the spatially arranged objects.

The solution is suitable for electronic computing devices and is marked by its computing economy.

INTRODUCTION

The stereoscopic evaluation of pairs of photographs using universal precision plotting instruments has become an established technique. These modern photogrammetric methods have by far their widest application in geodetic work, namely, in the triangulation of control points and in the compilation of topographic maps. Besides, in many scientific institutes and research laboratories, photogrammetric methods are in use which would provide a good approach to many different measuring problems. The remarkable advance in the development of powerful precision surveying lenses with relatively wide angle of view and the introduction of interchangeable glass plates of highest quality of flatness and emulsion have made the modern photogrammetric camera with its extreme inner stability another tool for precision measurements. The increasing use of large scale photographs by airborne cameras for precision surveying and similar application of such cameras on the ground and in the air for precision measurements in non-topographic fields have made it necessary to study the problem of orientation of a photogrammetric camera and the propagation of residual errors of the process of orientation. The error theory of stereo-photogrammetry has already been discussed by several authors [e.g. (5), (13)] and valuable detailed information on the subject has been published [e.g. (1), (2), (3)].* However, these studies are limited almost exclusively to the problem of determining the exterior orientation of two approximately vertical photographs by means of a universal plotting instrument. This specialization is consistent with the importance of this problem in the field of topographic photogrammetry. However, there is an indication that in the future theoretical photogrammetry must treat its problem not only in connection with certain types of evaluation machinery, but must continue analytical treatment where in the late twenties the optical mechanical reduction methods took over. At that time the numerical evaluation seemed too laborious to be considered for practical purposes. Today the development of an economical numerical solution of certain photogrammetric problems appears to be within reach with electronic computing devices.

The continuously increasing precision in the elements of photogrammetric cameras has made the camera the most precise component in the chain of steps leading from the photograph to the coordinate determination. Consequently, evaluation instrumentation must enable us to measure the plate coordinates of certain points - either for the purpose of orientation or final coordinate determination - with corresponding high accuracy. It is well known that the monocular or stereoscopic comparator designed according to Abbe's comparator principle, and used with precision grid plates for calibration, is an adequate plate reading device to meet precision requirements. However, the use of such an instrument calls for computation of the desired results from the measured plate coordinates by rather complex formulas and at high speed. In addition to the required automatic high speed computer an automatic comparator reading device, leading to an automatic computer input and an automatic data output, is required. In view of the possible development of small electronic computers together with other electronic recording devices, these requirements may possibly be met at a cost comparable to that of present day

* Bibliography at end of paper.

universal plotting machines. It is with these thoughts in mind that the analytical determination of the orientation of a photogrammetric camera is treated in detail, since this step is the common basis for all photogrammetric measuring methods. In addition to the use of independent photographs for special problems, the stereoscopic method may be considered as a combination of two independent cameras. The geometrical conditions as they exist between the two bundles of projective rays can be introduced analytically, by adding such condition equations in the process of the analytical reduction as are given by the functions expressing the relative orientation. Such an approach seems to be useful for analytical treatment and shows again the importance of the basic problem, namely, the orientation of the individual photogrammetric camera.

THE PHOTOGRAMMETRIC PROBLEM

In general, the purpose of photogrammetry may be defined as the determination of spatial coordinates of objects recorded on several photographs taken from different points. Consequently, we have the problems of the geometric conditions which exist for each individual photogrammetric camera as well as the geometric relations between such cameras. The projective geometrical conditions existing between pairs and triplets of photographs taken from different points are the fundamental relations for the process of triangulation. The nature of this triangulation step is the distinguishing factor in photogrammetric measuring methods. The division is made according to the manner in which the individual photographs are being combined in order to triangulate the single points of the model. An error theory concerning the final errors of the coordinates to be determined, or, in other words, the determination of model deformations, will be affected by the selected method. The manner of error propagation in the triangulation procedure depends upon whether the triangulation is performed by intersection or stereo-photogrammetry. It is also dependent upon whether the raw material is obtained by ground or airborne instrumentation. Finally, it is necessary to distinguish between a numerical and an optical-mechanical reduction. In case of the latter, the design characteristics of the evaluation instrument and the sequence of operations during the evaluation procedure will influence the error propagation. Consequently, the individual triangulation method applied to the same raw material may lead to different results depending on the reduction method used.

However different the theories of the triangulation procedures within the different fields of photogrammetry may appear, it is evident that all methods are based on the same raw material, namely, photographic records obtained from photogrammetric cameras. It seems justified, therefore, to define as the fundamental procedure for all photogrammetric problems, the orientation of photographs in such a way that each individual photograph will have the same orientation in space which it had during the moment of exposure. In this connection, it is immaterial whether the orientation of the photogrammetric camera is obtained from relative or absolute control points or by circle readings or other instrumental auxiliaries. It is further insignificant which geometrical reduction procedure is being

applied for the orientation and triangulation procedures. The actual photogrammetric problem is basically the same as any other measuring problem: that is, chiefly a problem of precision measurement in addition to one of exact mathematics. Independent of the triangulation method and evaluation procedure, the fundamental photogrammetric problem originates from the fact that during the reduction procedure the ideal orientation of the individual camera can only be approximated. The final coordinate determination will, therefore, be affected by the propagation of the individual camera errors of the interior and exterior orientation during the process of triangulation, in addition to the residual errors of the plate measurements of the image point under consideration.

In the following chapters a study is made of the orientation problem of an individual photogrammetric camera and the influence of the differential changes in the elements of orientation. Both the elements of interior and exterior orientation are considered and formulas are derived which deal with the different possibilities of orienting the axes of a photogrammetric camera with respect to a given coordinate system.

THE ORIENTATION OF A PHOTOGRAMMETRIC CAMERA[1]

(1) The principle of mathematical perspective.

The projective relation between two planes is expressed by the fractional linear equations: [e.g. (5)]

$$X = \frac{a_1\bar{x} + b_1\bar{y} + c_1}{a_o\bar{x} + b_o\bar{y} + 1}$$

and

$$Y = \frac{a_2\bar{x} + b_2\bar{y} + c_2}{a_o\bar{x} + b_o\bar{y} + 1} \qquad (1)$$

where

X, Y denote the Cartesian coordinates in one plane

$\bar{x}$, $\bar{y}$ the corresponding coordinates in the other plane

In equations (1) no preference is expressed for either of the two planes under consideration. Consequently, there must exist a second pair of formulas which express the reversed solution of formulas (1).

[1] The physical photogrammetric camera is idealized by assuming bundles of rays to be free of distortion and the image plane to be optically flat without emulsion shrinkage.

They are:

$$\bar{x} = \frac{a_1'X + b_1'Y + c_1'}{a_0'X + b_0'Y + 1}$$

and

$$\bar{y} = \frac{a_2'X + b_2'Y + c_2'}{a_0'X + b_0'Y + 1} \qquad (2)$$

Between the coefficients $a_o \ldots c_2$ and $a_o' \ldots c_2'$ there are the following relations:

$$a_o = \frac{a_2'b_o' - a_o'b_2'}{A'} \qquad\qquad a_o' = \frac{a_2b_o - a_ob_2}{A}$$

$$a_1 = \frac{b_2' - b_o'c_2'}{A'} \qquad\qquad a_1' = \frac{b_2 - b_oc_2}{A}$$

$$a_2 = \frac{a_o'c_2' - a_2'}{A'} \qquad\qquad a_2' = \frac{a_oc_2 - a_2}{A}$$

$$b_o = \frac{a_o'b_1' - a_1'b_o'}{A'} \qquad\qquad b_o' = \frac{a_ob_1 - a_1b_o}{A} \qquad (3)$$

$$b_1 = \frac{b_o'c_1' - b_1'}{A'} \qquad\qquad b_1' = \frac{b_oc_1 - b_1}{A}$$

$$b_2 = \frac{a_1' - a_o'c_1'}{A'} \qquad\qquad b_2' = \frac{a_1 - a_oc_1}{A}$$

$$c_1 = \frac{b_1'c_2' - b_2'c_1'}{A'} \qquad\qquad c_1' = \frac{b_1c_2 - b_2c_1}{A}$$

$$c_2 = \frac{a_2'c_1' - a_1'c_2'}{A'} \qquad\qquad c_2' = \frac{a_2c_1 - a_1c_2}{A}$$

where

$$A' = a_1'b_2' - a_2'b_1' \qquad\qquad A = a_1b_2 - a_2b_1$$

The complete symmetry and reversibility of the coefficients is the direct consequence of the reversibility of the two central perspectives involved.[2]

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[2] P. Tham has made some application of the reversibility of a central perspective in his paper. (12).

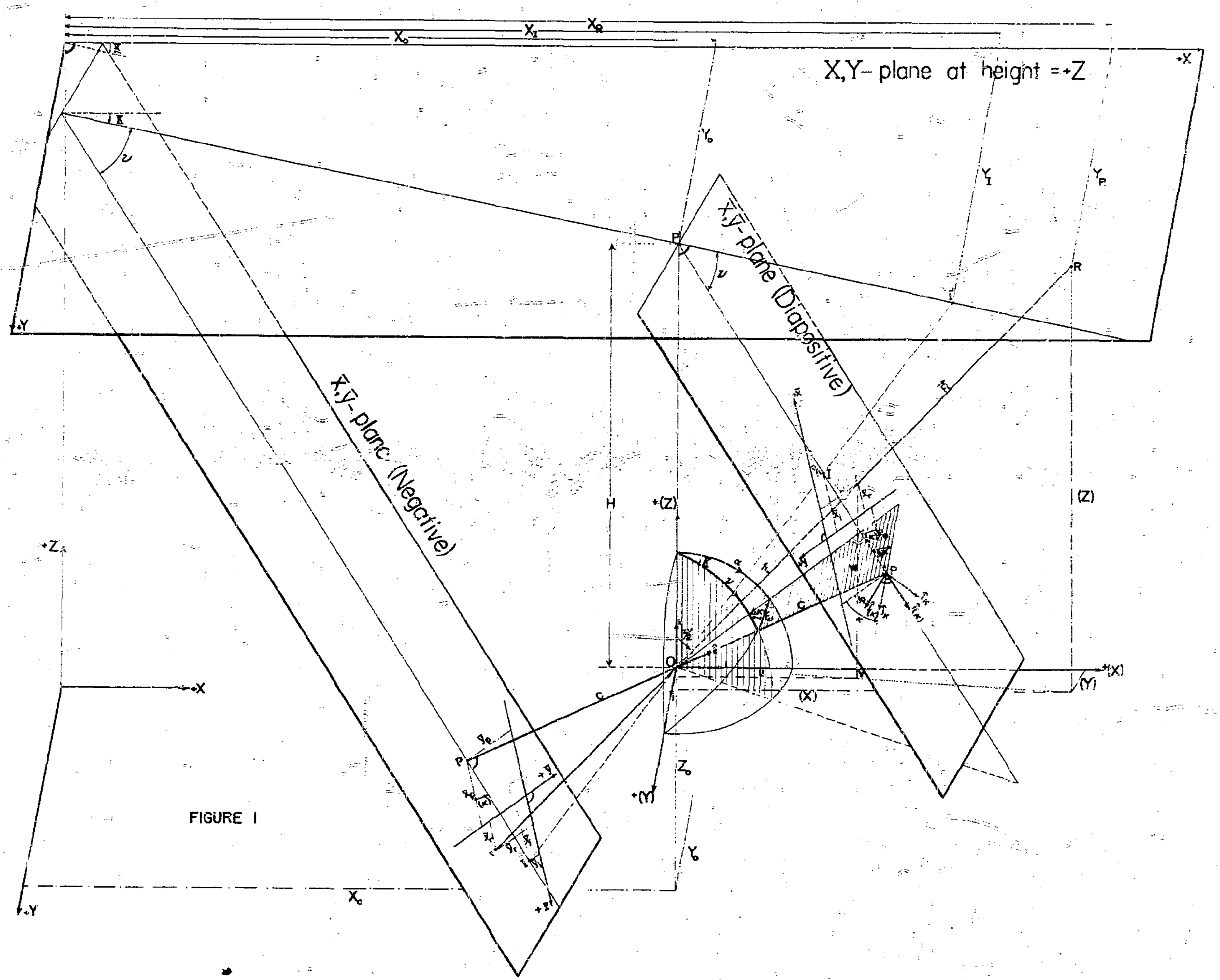
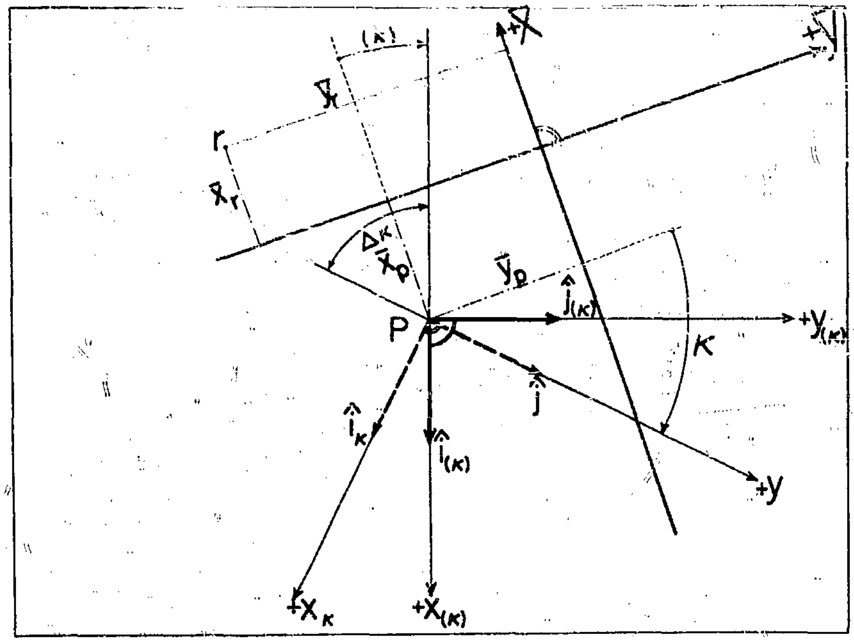

X,Y- plane at height =+Z
$\bar{x},\bar{y}$- plane (Diapositive)
$\bar{x},\bar{y}$- plane (Negative)
X_o
X_I
X_R
Y_o
Y_I
Y_P
+X
+Y
+Z
+(Z)
+(X)
(Y)
+(Y)
(Z)
(X)
H
Z_o
O
P
R

FIGURE I

$\bar{x},\bar{y}$-plane as Diapositive, seen from O



$\bar{x},\bar{y}$-plane as Negative, seen from O

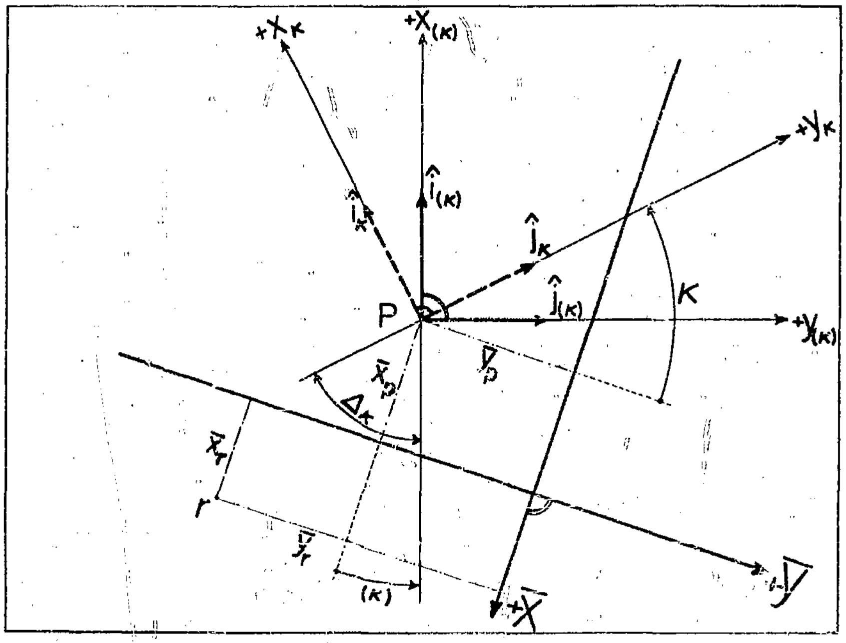


FIGURE 2

On the other hand the relations between the coordinates X, Y and $\bar{x}$, $\bar{y}$, respectively, can be expressed by the following functions:

$$X = f_1 \, (\bar{x}, \bar{y}, (\kappa), K, \nu, X_o, Y_o, H, \bar{x}_p, \bar{y}_p, c)$$

$$Y = f_2 \, (\bar{x}, \bar{y}, (\kappa), K, \nu, X_o, Y_o, H, \bar{x}_p, \bar{y}_p, c)$$

or (4)

$$X = f_3 \, (\bar{x}, \bar{y}, \kappa, \alpha, \omega, X_o, Y_o, H, \bar{x}_p, \bar{y}_p, c)$$

$$Y = f_4 \, (\bar{x}, \bar{y}, \kappa, \alpha, \omega, X_o, Y_o, H, \bar{x}_p, \bar{y}_p, c)$$

and the reversed relations:

$$\bar{x} = g_1 \, (X, Y, (\kappa), K, \nu, X_o, Y_o, H, \bar{x}_p, c)$$

$$\bar{y} = g_2 \, (X, Y, (\kappa), K, \nu, X_o, Y_o, H, \bar{y}_p, c)$$

or (5)

$$\bar{x} = g_3 \, (X, Y, \kappa, \alpha, \omega, X_o, Y_o, H, \bar{x}_p, c)$$

$$\bar{y} = g_4 \, (X, Y, \kappa, \alpha, \omega, X_o, Y_o, H, \bar{y}_p, c)$$

In formulas (4) and (5) we denote (See Figures 1 and 2):

X,Y the Cartesian coordinates in the plane at height Z; +Y is turned clockwise for 90° into +X as seen from the center of projection O.

$\bar{x}$,$\bar{y}$ the corresponding coordinates in the other plane (Negative or Diapositive); as seen from the center of projection, in the Negative +$\bar{y}$ is turned clockwise for 90° into +$\bar{x}$; and in the Diapositive +$\bar{y}$ is turned counterclockwise for 90° into +$\bar{x}$.

(κ) swing angle of the arbitrarily oriented $\bar{x}$, $\bar{y}$ - system against the direction of maximum tilt in the $\bar{x}$, $\bar{y}$ - plane

K swing angle of the arbitrarily oriented X, Y - system against the direction of maximum tilt in the X, Y - plane (primary rotation)

ν tilt angle between the two planes (secondary rotation)

α tilt angle component in the (X), (Z)-plane (primary rotation)

ω tilt angle component in the (Y), (Z)-plane tilted for α (secondary rotation)

κ swing angle of the arbitrarily oriented $\bar{x}$, $\bar{y}$ - system against the line of intersection of the α, ω - plane with the $\bar{x}$, $\bar{y}$-plane

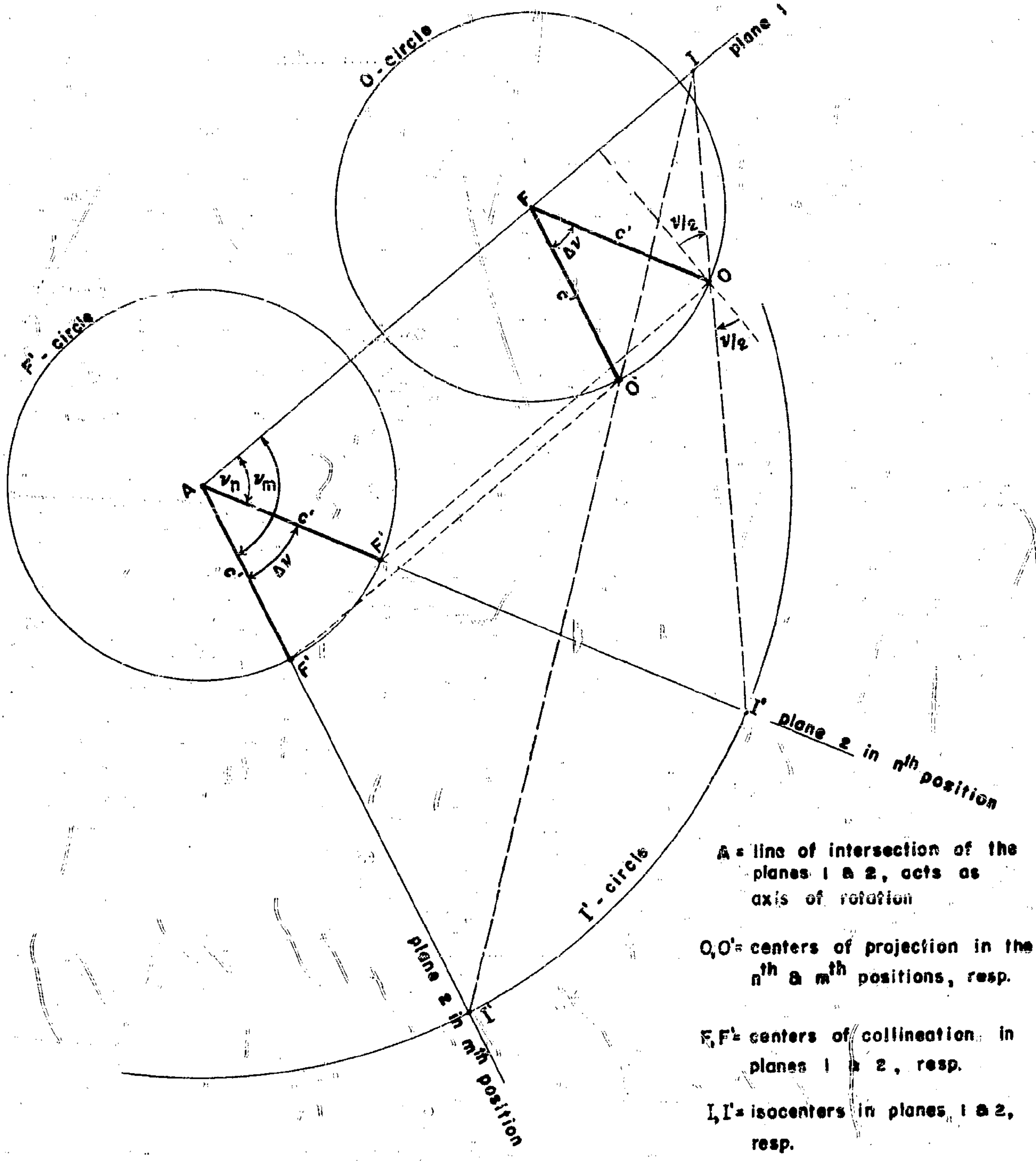

O - circle
plane 1
F' - circle
I' - circle
plane 2 in nth position
plane 2 in mth position
A = line of intersection of the planes 1 & 2, acts as axis of rotation
O, O' = centers of projection in the nth & mth positions, resp.
F, F' = centers of collineation in planes 1 & 2, resp.
I, I' = isocenters in planes 1 & 2, resp.

FIGURE 3.

X_o, Y_o, H relate the center of projection to the X, Y-plane at height Z

$\bar{x}_p$, $\bar{y}_p$, c relate the center of projection to the $\bar{x}$, $\bar{y}$-plane

$\bar{x}_p$ and $\bar{y}_p$ are the coordinates of the principal point P; c denotes the principal distance - or so called camera constant; $\bar{x}_p$, $\bar{y}_p$ and c are referred to as the elements of the interior orientation. The other parameters denote the elements of the exterior orientation.

A comparison of the formulas (1) and (2) with the corresponding formulas (4) and (5) shows that we have nine unknowns in the latter ones, namely

$$K,\ \nu,\ (\kappa),\ X_o,\ Y_o,\ H,\ \bar{x}_p,\ \bar{y}_p \text{ and } c$$

or

$$\alpha,\ \omega,\ \kappa\ ,\ X_o,\ Y_o,\ H,\ \bar{x}_p,\ \bar{y}_p \text{ and } c$$

and in the former ones only eight unknowns, denoted by the coefficients,

$$a_o,\ a_1,\ a_2,\ b_o,\ b_1,\ b_2,\ c_1 \text{ and } c_2$$

$$a_o',\ a_1',\ a_2',\ b_o',\ b_1',\ b_2',\ c_1' \text{ and } c_2'$$

Consequently, the relations expressed by formulas (1) and (2) do not give a unique solution but there are ∞^1 possibilities to bring the two planes in projective relation. This fact has its geometrical explanation in the set concerning the rotation of the center of collineation [(13) page 259]. Figure 3 shows this principle as it pertains to our case. In order to establish a unique relation between the two planes, additional information is needed. v. Gruber in (5) states that the unique solution can only be obtained if one of the elements of the interior orientation is given. In view of the reversibility of the problem under consideration, this statement can obviously be enlarged in as much as any one of the linear parameters $\bar{x}_p$, $\bar{y}_p$, c, X_o, Y_o, H is sufficient to make the solution unique. In addition the tilt angle ν is sufficient to fix the mutual situation of the two planes. From Figure 3 we see that the axis of rotation (A) and, consequently, the direction of maximum tilt in both planes is the same for all ∞^1 possible solutions. Consequently, the angles (κ) and K, which are defined as the swing angles of the arbitrarily oriented $\bar{x}$, $\bar{y}$ and X, Y - systems against the direction of maximum tilt in the corresponding planes, must be constant for all ∞^1 possible solutions and, therefore, must be functions of the coefficients only. v. Gruber in (5) gives the formula $\tan(\kappa) = a_o/b_o$ which leads, with our notations and formulas (3), to

$$\cot(\kappa) = \frac{a_o}{b_o} = \frac{a_2' b_o' - a_o' b_2'}{a_o' b_1' - a_1' b_o'} \tag{6}$$

and, as shown by P. Tham in (12), using the principle of reversibility, to

$$\cot K = \frac{a_o'}{b_o'} = \frac{a_2 b_o - a_o b_2}{a_o b_1 - a_1 b_o} \tag{7}$$

In addition we derive from Figure 1 the following formulas expressing the relation between the components of rotation:

$$\sin \omega = \sin \nu \sin K$$

$$\tan \alpha = \tan \nu \cos K$$

$$\sin \Delta\kappa = \sin \alpha \operatorname{cosec} \nu \quad \text{or} \quad \cot \Delta\kappa = \cos \nu \tan K = \sin \omega \cot \alpha$$

where $(\kappa) = \kappa = \Delta\kappa - 90^o$

and $\cos \nu = \cos \alpha \cos \omega$

$$\tan K = \tan \omega \operatorname{cosec} \alpha \tag{8}$$

Formulas (8) combined with formulas (6) and (7) show that instead of the tilt angle ν any one of the rotational components α, ω or κ similarly will be sufficient to determine a unique position of the two planes. It is now obvious that in case any one of the following variable parameters α, ω, κ, ν, $\bar{x}_p$, $\bar{y}_p$, c, X_o, Y_o, H is given the remaining parameters of the solution can be expressed as functions of the given parameter and the eight coefficients.

In photogrammetric problems the camera constant, denoted by c, would be the parameter most likely to be given or at least obtainable by an independent camera calibration.

Halonen has given in (7) two formulas which express, for the case that c is given, the H coordinate and the ν angle as functions of c and the coefficients. Using once more the principle of reversibility and formulas (3), we obtain from Halonen's formulas the modified and generalized relations:

$$H = c \cdot \frac{A}{A'} \left(\frac{a_o'^2 + b_o'^2}{a_o^2 + b_o^2} \right)^{3/2} \tag{9}$$

and

$$c = H \cdot \frac{A'}{A} \left(\frac{a_o^2 + b_o^2}{a_o'^2 + b_o'^2} \right)^{3/2} \tag{10}$$

and

$$\sin \nu = \frac{c}{A'} \cdot \frac{a_o'^2 + b_o'^2}{\sqrt{a_o^2 + b_o^2}} = \frac{H}{A} \cdot \frac{a_o^2 + b_o^2}{\sqrt{a_o'^2 + b_o'^2}} \tag{11}$$

where A and A' again have the meaning as used in formulas (3). The remaining four unknown parameters are the coordinates of the center of projection projected into the X, Y - plane and into the $\bar{x}$, $\bar{y}$ - plane, denoted by X_o, Y_o and $\bar{x}_p$, $\bar{y}_p$, respectively.

From Figure 1 we read

$$\begin{aligned} X_o &= X_i - H \tan \frac{\nu}{2} \cos K \\ Y_o &= Y_i - H \tan \frac{\nu}{2} \sin K \end{aligned} \tag{12}$$

and again from Figure 1 or by using the principle of reversibility

$$\begin{aligned} \bar{x}_p &= \bar{x}_i - c \tan \frac{\nu}{2} \cos (\kappa) \\ \bar{y}_p &= \bar{y}_i - c \tan \frac{\nu}{2} \sin (\kappa) \end{aligned} \tag{13}$$

In formulas (12) and (13) X_i, Y_i and $\bar{x}_i$, $\bar{y}_i$ denote the coordinates of the isocenters I' and I in the X, Y-plane and in the $\bar{x}$, $\bar{y}$-plane, respectively. As may be seen from Figure 3 the isocenters remain constant for all ∞^1 possible arrangements and, consequently, the coordinates of these points are functions of the coefficients only. v. Gruber in (5) gives for X_i, Y_i expressions which are in our notation

$$X_i = \frac{a_o(a_1 + b_2) - b_o(a_2 - b_1)}{a_o^2 + b_o^2} \qquad Y_i = \frac{a_o(a_2 - b_1) + b_o(a_1 + b_2)}{a_o^2 + b_o^2}$$

and, consequently, the reversed relations are (14)

$$\bar{x}_i = \frac{a_o'(a_1' + b_2') - b_o'(a_2' - b_1')}{a_o'^2 + b_o'^2} \qquad \bar{y}_i = \frac{a_o'(a_2' - b_1') + b_o'(a_1' + b_2')}{a_o'^2 + b_o'^2}$$

Applying formulas (3) to (14), we obtain from (12) and (13) after suitable transformations and using again the principle of reversibility:

$$X_o = \frac{a_o a_1 + b_o b_1}{a_o^2 + b_o^2} + a_o' \sqrt{\frac{A^2}{(a_o^2 + b_o^2)^2} - \frac{H^2}{a_o'^2 + b_o'^2}}$$

$$Y_o = \frac{a_o a_2 + b_o b_2}{a_o^2 + b_o^2} + b_o' \sqrt{\frac{A^2}{(a_o^2 + b_o^2)^2} - \frac{H^2}{a_o'^2 + b_o'^2}} \tag{15}$$

$$\bar{x}_p = \frac{a_o' a_1' + b_o' b_1'}{a_o'^2 + b_o'^2} + a_o \sqrt{\frac{A'^2}{(a_o'^2 + b_o'^2)^2} - \frac{c^2}{a_o^2 + b_o^2}}$$

$$\bar{y}_p = \frac{a_o' a_2' + b_o' b_2'}{a_o'^2 + b_o'^2} + b_o \sqrt{\frac{A'^2}{(a_o'^2 + b_o'^2)^2} - \frac{c^2}{a_o^2 + b_o^2}} \tag{16}$$

By means of the derived formulas all elements of orientation may now be determined, namely (κ) and K are obtained from formulas (6) and (7), H from formula (9), ν from formula (11), its rotational components α, ω, κ from formulas (8) and the remaining parameters X_o, Y_o, $\bar{x}_p$, and $\bar{y}_p$ from formulas (15) and (16).

Despite the claim of some authors[3] that the prescribed approach gives a method of maximum accuracy for the determination of the elements of orientation, these formulas have a serious limitation. Not only are the formulas correct only for the mutual situation of two planes, but they will fail in the case that the tilt angle ν becomes zero or 180°. In practical photogrammetry, however, this case is the most important one for aerial problems. Moreover, it occurs in special cases of ground photogrammetry as well. Even though in such cases $\nu \neq 0^\circ$ or 180°, the tilt angle may approach these values, so that the numerical computations with the above derived formulas become unreliable.

To study this problem further we will first express the coefficients as functions of the elements of orientation. Omitting the derivation given in the next chapter of this paper, we obtain from formulas (45) to (48) by comparison with the formulas (1) and (2) , for the case of two planes:

(a) for the unprimed values:

$$a_o = \frac{\cos(\kappa)\tan\nu}{D} = \frac{\tan\omega\sin\kappa + \tan\alpha\sec\omega\cos\kappa}{D}$$

$$b_o = \frac{\sin(\kappa)\tan\nu}{D} = \frac{-\tan\omega\cos\kappa + \tan\alpha\sec\omega\sin\kappa}{D} \tag{17}$$

$$a_1 = \frac{Z-Z_o}{D}\left(-\cos(\kappa)\cos K + \sin(\kappa)\sin K\sec\nu + \frac{X_o}{Z-Z_o}\cos(\kappa)\tan\nu\right)$$

$$= \frac{Z-Z_o}{D}\left[\sin\kappa\tan\alpha\tan\omega - \cos\kappa\sec\omega + \frac{X_o}{Z-Z_o}(\sin\kappa\tan\omega + \cos\kappa\tan\alpha\sec\omega)\right]$$

---

[3] e.g. R. S. Halonen in (7) pages 14 and 18.

$$b_1 = \frac{Z-Z_o}{D}\left(-\sin(\kappa)\cos K - \cos(\kappa)\sin K \sec\nu + \frac{X_o}{Z-Z_o}\sin(\kappa)\tan\nu\right)$$

$$= \frac{Z-Z_o}{D}\Big[-\cos\kappa\tan\omega\tan\alpha - \sin\kappa\sec\omega$$

$$+ \frac{X_o}{Z-Z_o}(-\cos\kappa\tan\omega + \sin\kappa\tan\alpha\sec\omega)\Big]$$

$$c_1 = \frac{Z-Z_o}{D}\Big[c\left(\tan\nu\cos K + \frac{X_o}{Z-Z_o}\right)$$

$$+ \bar{x}_p\left(\cos(\kappa)\cos K - \sin(\kappa)\sin K\sec\nu - \frac{X_o}{Z-Z_o}\cos(\kappa)\tan\nu\right)$$

$$+ \bar{y}_p\left(\sin(\kappa)\cos K + \cos(\kappa)\sin K\sec\nu - \frac{X_o}{Z-Z_o}\sin(\kappa)\tan\nu\right)\Big]$$

$$= \frac{Z-Z_o}{D}\Big\{c\left(\tan\alpha + \frac{X_o}{Z-Z_o}\right)$$

$$+ \bar{x}_p\left[-\sin\kappa\tan\alpha\tan\omega + \cos\kappa\sec\omega - \frac{X_o}{Z-Z_o}(\sin\kappa\tan\omega + \cos\kappa\tan\alpha\sec\omega)\right]$$

$$+ \bar{y}_p\left[\cos\kappa\tan\alpha\tan\omega + \sin\kappa\sec\omega - \frac{X_o}{Z-Z_o}(-\cos\kappa\tan\omega + \sin\kappa\tan\alpha\sec\omega)\right]\Big\}$$

$$a_2 = \frac{Z-Z_o}{D}\left(-\cos(\kappa)\sin K - \sin(\kappa)\cos K\sec\nu + \frac{Y_o}{Z-Z_o}\cos(\kappa)\tan\nu\right)$$

$$= \frac{Z-Z_o}{D}\left[-\sin\kappa\sec\alpha + \frac{Y_o}{Z-Z_o}(\sin\kappa\tan\omega + \cos\kappa\tan\alpha\sec\omega)\right] \qquad (17)\ \text{cont.}$$

$$b_2 = \frac{Z-Z_o}{D}\left(-\sin(\kappa)\sin K + \cos(\kappa)\cos K\sec\nu + \frac{Y_o}{Z-Z_o}\sin(\kappa)\tan\nu\right)$$

$$= \frac{Z-Z_o}{D}\left[\cos\kappa\sec\alpha + \frac{Y_o}{Z-Z_o}(-\cos\kappa\tan\omega + \sin\kappa\tan\alpha\sec\omega)\right]$$

$$c_2 = \frac{Z-Z_o}{D}\Big[c\left(\tan\nu\sin K + \frac{Y_o}{Z-Z_o}\right)$$

$$+ \bar{x}_p\left(\cos(\kappa)\sin K + \sin(\kappa)\cos K\sec\nu - \frac{Y_o}{Z-Z_o}\cos(\kappa)\tan\nu\right)$$

$$+ \bar{y}_p\left(\sin(\kappa)\sin K - \cos(\kappa)\cos K\sec\nu - \frac{Y_o}{Z-Z_o}\sin(\kappa)\tan\nu\right)\Big]$$

$$c_2 = \frac{Z-Z_o}{D} \left\{ c(\tan\omega \sec\alpha + \frac{Y_o}{Z-Z_o}) \right.$$

$$+ \bar{x}_p \left[\sin\kappa \sec\alpha - \frac{Y_o}{Z-Z_o} (\sin\kappa \tan\omega + \cos\kappa \tan\alpha \sec\omega) \right] \qquad (17) \text{ cont.}$$

$$\left. + \bar{y}_p \left[-\cos\kappa \sec\alpha - \frac{Y_o}{Z-Z_o} (-\cos\kappa \tan\omega + \sin\kappa \tan\alpha \sec\omega) \right] \right\}$$

where

$$D = c - \tan\nu \left[\bar{x}_p \cos(\kappa) + \bar{y}_p \sin(\kappa) \right]$$

$$= c - \bar{x}_p (\tan\omega \sin\kappa + \tan\alpha \sec\omega \cos\kappa)$$

$$+ \bar{y}_p (\tan\omega \cos\kappa - \tan\alpha \sec\omega \sin\kappa)$$

and (b) for the primed values:

$$a_o' = \frac{\cos K \tan\nu}{D'} = \frac{\tan\alpha}{D'}$$

$$b_o' = \frac{\sin K \tan\nu}{D'} = \frac{\tan\omega \sec\alpha}{D'}$$

$$a_1' = \frac{c}{D'} (-\cos K \cos(\kappa) + \sin K \sin(\kappa) \sec\nu + \frac{\bar{x}_p}{c} \cos K \tan\nu)$$

$$= \frac{c}{D'} (-\sec\omega \cos\kappa + \tan\alpha \tan\omega \sin\kappa + \frac{\bar{x}_p}{c} \tan\alpha) \qquad (18)$$

$$b_1' = \frac{c}{D'} (-\cos K \sin(\kappa) \sec\nu - \sin K \cos(\kappa) + \frac{\bar{x}_p}{c} \sin K \tan\nu)$$

$$= \frac{c}{D'} (-\sec\alpha \sin\kappa + \frac{\bar{x}_p}{c} \tan\omega \sec\alpha)$$

$$c_1' = \frac{c}{D'} \left[X_o (\cos K \cos(\kappa) - \sin K \sin(\kappa) \sec\nu - \frac{\bar{x}_p}{c} \cos K \tan\nu) \right.$$

$$+ Y_o (\sin K \cos(\kappa) + \cos K \sin(\kappa) \sec\nu - \frac{\bar{x}_p}{c} \sin K \tan\nu)$$

$$\left. + (Z-Z_o)(\tan\nu \cos(\kappa) + \frac{\bar{x}_p}{c}) \right]$$

$$c_1' = \frac{c}{D'}\Big[X_o(\sec\omega\cos\kappa - \tan\alpha\tan\omega\sin\kappa - \frac{\bar{x}_p}{c}\tan\alpha)$$

$$+ Y_o(\sec\alpha\sin\kappa - \frac{\bar{x}_p}{c}\tan\omega\sec\alpha)$$

$$+ (Z-Z_o)(\tan\alpha\sec\omega\cos\kappa + \tan\omega\sin\kappa + \frac{\bar{x}_p}{c})\Big]$$

$$a_2' = \frac{c}{D'}(-\sin K\cos(\kappa)\sec\nu - \cos K\sin(\kappa) + \frac{\bar{y}_p}{c}\cos K\tan\nu)$$

$$= \frac{c}{D'}(-\tan\alpha\tan\omega\cos\kappa - \sec\omega\sin\kappa + \frac{\bar{y}_p}{c}\tan\alpha)$$

$$b_2' = \frac{c}{D'}(\cos K\cos(\kappa)\sec\nu - \sin K\sin(\kappa) + \frac{\bar{y}_p}{c}\sin K\tan\nu)$$

$$= \frac{c}{D'}(\sec\alpha\cos\kappa + \frac{\bar{y}_p}{c}\tan\omega\sec\alpha) \qquad (18)\text{cont.}$$

$$c_2' = \frac{c}{D'}\Big[X_o(\cos K\sin(\kappa) + \sin K\cos(\kappa)\sec\nu - \frac{\bar{y}_p}{c}\cos K\tan\nu)$$

$$+ Y_o(-\cos K\cos(\kappa)\sec\nu + \sin K\sin(\kappa) - \frac{\bar{y}_p}{c}\sin K\tan\nu)$$

$$+ (Z-Z_o)(\tan\nu\sin(\kappa) + \frac{\bar{y}_p}{c})\Big]$$

$$= \frac{c}{D'}\Big[X_o(\tan\alpha\tan\omega\cos\kappa + \sec\omega\sin\kappa - \frac{\bar{y}_p}{c}\tan\alpha)$$

$$+ Y_o(-\sec\alpha\cos\kappa - \frac{\bar{y}_p}{c}\tan\omega\sec\alpha)$$

$$+ (Z-Z_o)(-\tan\omega\cos\kappa + \tan\alpha\sec\omega\sin\kappa + \frac{\bar{y}_p}{c})\Big]$$

where

$$D' = (Z-Z_o) - X_o\tan\nu\cos K - Y_o\tan\nu\sin K$$

$$= (Z-Z_o) - X_o\tan\alpha - Y_o\tan\omega\sec\alpha$$

For ν approaching zero or 180^o, or in other words, α approaching zero or 180^o and ω approaching zero, we see that a_o, b_o, a_o' and b_o' approach zero. Formulas (6) and (7) render undetermined expressions for (κ) and K of the form $\frac{0}{0}$, which is obviously explained by the fact that for ν = zero

or 180° there is no longer a defined direction of maximum tilt and, consequently, the swing angles (κ) and K become meaningless. In the same way the formulas (9), (10), (11), (15) and (16) fail to give a determinate answer. These facts lead to two conclusions.

First, it is necessary to present the orientation problem in such a way that the parameters involved can be so arranged that for the majority of practical cases they maintain their significance independent of the geometrical configuration. Such an approach will not only help to simplify the problem but must be considered as necessary for a general analytical solution. We satisfy this requirement by relating the orientation problem to such rotations as expressed by the α, ω, κ -system. The relations to the ν, K, (κ) -system, if desired, are given by formulas (8), geometrical conditions permitting. Problems of aerial photogrammetry and special problems of ground photogrammetry (for instance, trajectory measurements where the optical axes are within a tilt ν from 0° to about 45°) can always be expressed in the α, ω -system. There are a few cases of ground photogrammetry so characterized that for tilt angles ν close to 90° the optical axis may point close to the direction of $\pm$ Y-axis ($\mathrm{K} \sim 90°$ of 270°). In such cases α becomes ambiguous. This difficulty, however, can easily be eliminated by first rotating the original XYZ-system 90° or 270° about the Z-axis.

Second, we express the orientation elements as such functions of c and the coefficients that the formulas do not render undetermined values in case a_o, b_o, a'_o and b'_o become zero. From formulas (17) and (18) in connection with formulas (3), the following expressions may be obtained:

$$X_o = \frac{c^2B^2(a_oa_1 + b_ob_1) + B(a_1\bar{x}_p + b_1\bar{y}_p + c_1)}{c^2B^2(a_o^2 + b_o^2) + 1} \tag{19}$$

$$Y_o = \frac{c^2B^2(a_oa_2 + b_ob_2) + B(a_2\bar{x}_p + b_2\bar{y}_p + c_2)}{c^2B^2(a_o^2 + b_o^2) + 1}$$

where

$$B = \frac{1}{1 + a_o\bar{x}_p + b_o\bar{y}_p}$$

and

$$\bar{x}_p = \frac{H^2B'^2(a'_oa'_1 + b'_ob'_1) + B'(a'_1X_o + b'_1Y_o + c'_1)}{H^2B'^2(a'^2_o + b'^2_o) + 1} \tag{20}$$

$$\bar{y}_p = \frac{H^2B'^2(a'_oa'_2 + b'_ob'_2) + B'(a'_2X_o + b'_2Y_o + c'_2)}{H^2B'^2(a'^2_o + b'^2_o) + 1}$$

where

$$B' = \frac{1}{1 + a_o'X_o + b_o'Y_o}$$

Further:

$$H = c\left(\frac{A \quad B^3}{A' \quad B'^3}\right)^{1/4} \tag{21}$$

and

$$c = H\left(\frac{A' \quad B'^3}{A \quad B^3}\right)^{1/4} \tag{22}$$

where

$$AA' = (a_1b_2 - a_2b_1)(a_1'b_2' - a_2'b_1') = (1+a_oc_1'+b_oc_2') = (1 + a_o'c_1 + b_o'c_2)$$

$$\tan \alpha = a_o'HB' \tag{23}$$

$$\tan \omega = b_o'HB' \cos \alpha = b_o' \left[\frac{1}{(B'H)^2} + a_o'^2\right]^{-1/2} \tag{24}$$

$$\tan \kappa = \frac{a_oY_o - a_2}{b_2 - b_oY_o} \tag{25}$$

For X_o, Y_o and $\bar{x}_p$, $\bar{y}_p$, respectively, formulas (15) and (16) cannot be changed simply because for $\alpha = 0^\circ$ or 180° and $\omega = 0^\circ$, not only a_o, b_o, a_o' and b_o' become zero, but $a_1 \equiv b_2$, $a_1' \equiv b_2'$ and $b_1 \equiv a_2$, $b_1' \equiv a_2'$. Hence, there are only four independent quantities in formulas (1) and (2), i.e. the solution is ∞^3 undetermined. Obviously, therefore, the center of projection may have any spatial position. For c given, H is determined as may be seen from formula (21) since B and B' become 1. Formulas (19) and (20) show that in such a case two further linear parameters must be given, e.g. $\bar{x}_p$ and $\bar{y}_p$, in order to fix the position of the center of projection in space. This fact must be given serious consideration in the computation of the spatial resection problem from vertical photographs. The computed elements of orientation will have physical significance only if the elements of the interior orientation, denoted by c, $\bar{x}_p$ and $\bar{y}_p$, are independently determined from a camera calibration. Hence, in such cases we deal with six degrees of freedom only. Consequently, the relations expressed in formulas (1) and (2) are not independent. Two condition equations must exist between the coefficients. The same problem exists in

the case of ground photogrammetry[4] where the position of the center of projection X_o, Y_o, Z_o is usually independently determined by geodetic measurements in the system of the control points. Hence, again only six degrees of freedom remain. Again the relations expressed in formulas (1) and (2) are not independent. The condition equations for the unprimed values are:

$$
\begin{aligned}
& a_o b_o + a_1 b_1 + a_2 b_2 = 0 \\
& a_o^2 + a_1^2 + a_2^2 - b_o^2 - b_1^2 - b_2^2 = 0
\end{aligned}
\tag{26}
$$

and for the primed values:

$$
\begin{aligned}
& a_1' a_2' + b_1' b_2' + c_1' c_2' = 0 \\
& a_1'^2 + b_1'^2 + c_1'^2 - a_2'^2 - b_2'^2 - c_2'^2 = 0
\end{aligned}
\tag{27}
$$

These condition equations must be included in any analytical solution, either unique with three points or overdetermined with n-points ($n > 3$). The unknown parameters may be expressed from formulas (19) - (25). In the case of ground photogrammetry we may arrange the reference coordinate system so that $X_o = Y_o = 0$ and $H = 1$. Thus we obtain the following relations:

$$\bar{x}_p = \frac{a_o' a_1' + b_o' b_1' + c_1'}{a_o'^2 + b_o'^2 + 1} \tag{28}$$

$$\bar{y}_p = \frac{a_o' a_2' + b_o' b_2' + c_2'}{a_o'^2 + b_o'^2 + 1} \tag{29}$$

$$
\begin{aligned}
c &= \left[\frac{(a_1' b_2' - a_2' b_1')(1 + a_o \bar{x}_p + b_o \bar{y}_p)^3}{(a_1 b_2 - a_2 b_1)}\right]^{1/4} \\
&= \left[\frac{(a_1' b_2' - a_2' b_1')^2 (1 + a_o \bar{x}_p + b_o \bar{y}_p)^3}{(1 + a_o c_1' + b_o c_2')}\right]^{1/4}
\end{aligned}
\tag{30}
$$

$$\tan \alpha = a_o' \tag{31}$$

$$\tan \omega = b_o' (a_o'^2 + 1)^{-1/2} \tag{32}$$

---

4 Compare (11).

$$\tan\kappa = \frac{-a_2}{+b_2} = \frac{a_o' c_2' - a_2'}{a_o' c_1' - a_1'} \qquad (33)$$

Formulas (28) to (33) are especially suitable for the calibration of a photogrammetric camera, e.g. by means of star photographs.

(2) The principle of mathematical projection.

The limitations of the treatment as given to the orientation problem of a photogrammetric camera in the preceding paragraphs are obvious. The assumption that the problem is concerned with the mutual situation of two planes is in general justified only in special cases of non-topographic photogrammetry, e.g. an ideal case is present when stars are taken as control points. The solution with plate constants is quite attractive for the analytical approach. However, such a solution loses much of its usefulness if, as generally in ground and aerial photogrammetric problems, the existing condition equations must be introduced. It becomes almost worthless if additional parameters of the solution are given, e.g. from instrumental readings in the form of dial or level settings. Further, it will be shown that the analytical orientation by a rigorous least squares adjustment is by no means more complicated if the elements of orientation are used as unknowns instead of the plate constants. On the contrary, the solution loses its complexity and offers the possibility to compute in the most economical way, by allowing the reduction of the number of unknown parameters by simply setting the corrections Δ for the given parameters to zero. The computation of approximate values for the unknowns of the solution is simpler if quantities are used for the unknowns which have easily defined physical meanings and which may become directly available from instrument readings or calibration procedures. For a rigorous least squares adjustment, approximate values become necessary in both solutions (plate constants or orientation elements) because the observational equations are always non-linear. Considering the general case, we must remember that the determination of all nine unknowns cannot be made if the control points are located in a plane, regardless of the number of such points available for the solution. In general, we may say that the bundle of rays which is being reconstructed by an analytical or geometrical process will only be close to the original bundle within the limits of the space defined by the control points used in the reconstruction. In case we intend to compute the actual physical orientation elements of the camera, we must either determine at least one of the nine mentioned parameters independently with sufficient accuracy or else the control points must be sufficiently spread in all three coordinates.

We introduce now a spatial rectangular Cartesian coordinate system - (X), (Y), (Z) - in such a way that its origin is the center of projection and its orientation is consistent with an arbitrarily chosen XYZ-system. Further, we consider the interior and exterior orientation of a photograph simultaneously, that is, we consider the photograph in its spatial position. Consequently, the plate coordinate system $\bar{x}$, $\bar{y}$ will be introduced as another rectangular spatial system with its origin

also at the center of projection.[5] The plate is now introduced as a diapositive. (Figure 1.)

O is the center of projection. Its coordinates are denoted by X_o, Y_o, Z_o. In O, we establish two rectangular Cartesian systems. The (X), (Y), (Z)-system, denoted as the station system, is expressed by the unit vectors, i, j, k. In this system, a spatial target point R is determined by the coordinates (X), (Y), (Z). Its image point r is denoted by the coordinates u, v, w, respectively. The other Cartesian system expressed by the unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ denotes the camera system. Its orientation, relative to the station system, is expressed by three elements of exterior orientation denoted by two position angles and by the swing angle of the fiducial mark system $(\bar{x},\bar{y})$. The two position angles are K and ν or α and ω, depending on the arrangement of the rotation axes. The corresponding swing angles are denoted by (κ) and κ respectively. (Compare Figure 1 and 2). The length of the plate perpendicular - the principal distance or camera constant-is denoted by c. $\bar{x}_p$ and $\bar{y}_p$ are the coordinates of the principal point P in any rectangular plane coordinate system established by fiducial marks and denoted by $\bar{x}$ and $\bar{y}$. x and y are the plane plate coordinates of the image point r in the oriented xy-system whose origin is in the principal point P.

Further, we introduce the image vector $\vec{r}$ and the object vector $\vec{R}$, which are expressed as follows:

$$\vec{r} = iu + jv + kw = \hat{i}x + \hat{j}y + \hat{k}c \tag{34}$$

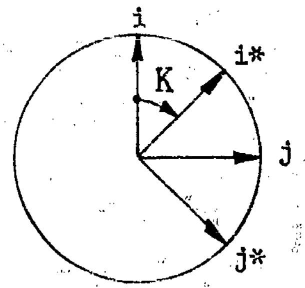
and $\vec{R} = i(X) + j(Y) + k(Z)$

Formulas (34) express the entire orientation problem of an individual photogrammetric camera. For the practical application it is only necessary to derive expressions which will transform the triple vectors i, j, k into the triple vectors $\hat{i}$, $\hat{j}$, $\hat{k}$, and vice versa. The transformation matrix which exists between the two vector triples may be written as

$$\begin{array}{c|c} & \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \end{array} \\ \hline \begin{array}{c} i \\ j \\ k \end{array} & \begin{pmatrix} i\hat{i} & i\hat{j} & i\hat{k} \\ j\hat{i} & j\hat{j} & j\hat{k} \\ k\hat{i} & k\hat{j} & k\hat{k} \end{pmatrix} \end{array} \tag{35}$$

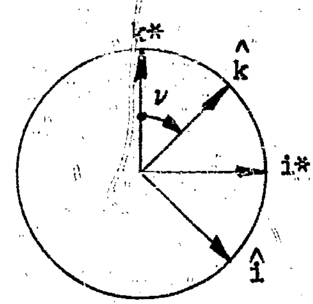
5. Compare (8).

The individual transformation matrices are: for a rotation in K, denoted temporarily by an asterisk (*),



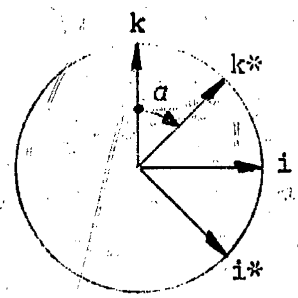
$$\begin{array}{c|c} & \begin{array}{ccc} i^* & j^* & k^* \end{array} \\ \hline \begin{array}{c} i \\ j \\ k \end{array} & \begin{pmatrix} +\cos K & -\sin K & 0 \\ +\sin K & +\cos K & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

for a rotation in ν-tilt,



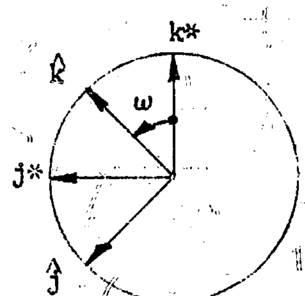
$$\begin{array}{c|c} & \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \end{array} \\ \hline \begin{array}{c} i^* \\ j^* \\ k^* \end{array} & \begin{pmatrix} +\cos \nu & 0 & +\sin \nu \\ 0 & 1 & 0 \\ -\sin \nu & 0 & +\cos \nu \end{pmatrix} \end{array} \quad (36a)$$

for a rotation in α-tilt denoted temporarily by an asterisk (*),



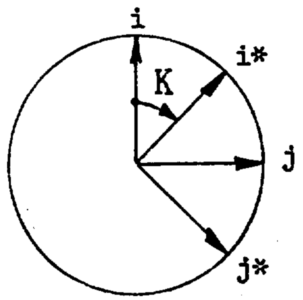
$$\begin{array}{c|c} & \begin{array}{ccc} i^* & j^* & k^* \end{array} \\ \hline \begin{array}{c} i \\ j \\ k \end{array} & \begin{pmatrix} +\cos \alpha & 0 & +\sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & +\cos \alpha \end{pmatrix} \end{array}$$

for a rotation in ω-tilt,



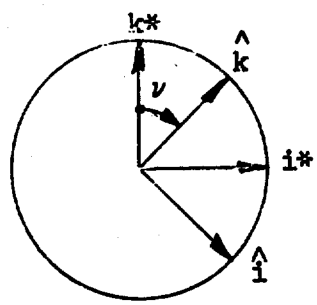
$$\begin{array}{c|c} & \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \end{array} \\ \hline \begin{array}{c} i^* \\ j^* \\ k^* \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & +\cos \omega & +\sin \omega \\ 0 & -\sin \omega & +\cos \omega \end{pmatrix} \end{array} \quad (36b)$$

The individual transformation matrices are: for a rotation in K, denoted temporarily by an asterisk (*),



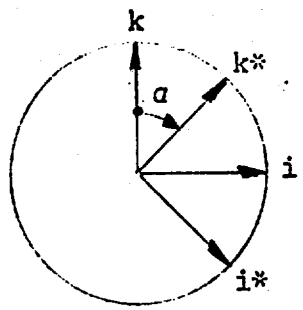
$$\begin{array}{c|c} & \begin{array}{ccc} i^* & \quad j^* & \quad k^* \end{array} \\ \hline \begin{array}{c} i \\ j \\ k \end{array} & \begin{pmatrix} +\cos K & -\sin K & 0 \\ +\sin K & +\cos K & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

for a rotation in ν-tilt,



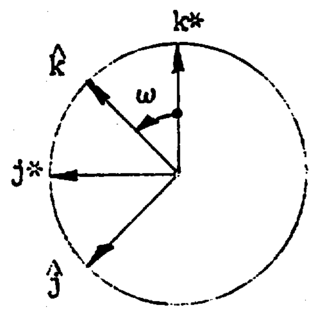
$$\begin{array}{c|c} & \begin{array}{ccc} \hat{i} & \quad \hat{j} & \quad \hat{k} \end{array} \\ \hline \begin{array}{c} i^* \\ j^* \\ k^* \end{array} & \begin{pmatrix} +\cos \nu & 0 & +\sin \nu \\ 0 & 1 & 0 \\ -\sin \nu & 0 & +\cos \nu \end{pmatrix} \end{array} \tag{36a}$$

for a rotation in α-tilt denoted temporarily by an asterisk (*),



$$\begin{array}{c|c} & \begin{array}{ccc} i^* & \quad j^* & \quad k^* \end{array} \\ \hline \begin{array}{c} i \\ j \\ k \end{array} & \begin{pmatrix} +\cos \alpha & 0 & +\sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & +\cos \alpha \end{pmatrix} \end{array}$$

for a rotation in ω-tilt,



$$\begin{array}{c|c} & \begin{array}{ccc} \hat{i} & \quad \hat{j} & \quad \hat{k} \end{array} \\ \hline \begin{array}{c} i^* \\ j^* \\ k^* \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & +\cos \omega & +\sin \omega \\ 0 & -\sin \omega & +\cos \omega \end{pmatrix} \end{array} \tag{36b}$$

The combined transformation matrices are: in the ν, K - system:

$$\begin{array}{c|ccc} & \hat{i} & \hat{j} & \hat{k} \\ \hline i & & & \\ j & & & \\ k & & & \end{array}\!\!\!\!\!\!\!\!\!\!\begin{pmatrix} +\cos K \cos\nu & -\sin K & +\cos K \sin\nu \\ +\sin K \cos\nu & +\cos K & +\sin K \sin\nu \\ -\sin\nu & 0 & +\cos\nu \end{pmatrix}$$

in the α, ω - system:

$$\begin{array}{c|c} & \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \end{array} \\ \hline \begin{array}{c} i \\ j \\ k \end{array} & \begin{pmatrix} +\cos\alpha & -\sin\alpha \sin\omega & +\sin\alpha \cos\omega \\ 0 & +\cos\omega & +\sin\omega \\ -\sin\alpha & -\cos\alpha \sin\omega & +\cos\alpha \cos\omega \end{pmatrix} \end{array}$$

or in the K, ν - system:

$$\hat{i} = +i \cos K \cos\nu + j \sin K \cos\nu - k \sin\nu$$
$$\hat{j} = -i \sin K + j \cos K \quad 0$$
$$\hat{k} = +i \cos K \sin\nu + j \sin K \sin\nu + k \cos\nu$$

in the α, ω - system:

$$\hat{i} = +i \cos\alpha \quad 0 \quad - k \sin\alpha$$
$$\hat{j} = -i \sin\alpha \sin\omega + j \cos\omega - k \cos\alpha \sin\omega$$
$$\hat{k} = +i \sin\alpha \cos\omega + j \sin\omega + k \cos\alpha \cos\omega$$

The reversed relations are: (37)

$$i = +\hat{i} \cos K \cos\nu - \hat{j} \sin K + \hat{k} \cos K \sin\nu$$
$$j = +\hat{i} \sin K \cos\nu + \hat{j} \cos K + \hat{k} \sin K \sin\nu$$
$$k = -\hat{i} \sin\nu \quad 0 \quad + \hat{k} \cos\nu$$

$$i = +\hat{i} \cos\alpha - \hat{j} \sin\alpha \sin\omega + \hat{k} \sin\alpha \cos\omega$$
$$j = \quad 0 \quad + \hat{j} \cos\omega + \hat{k} \sin\omega$$
$$k = -\hat{i} \sin\alpha - \hat{j} \cos\alpha \sin\omega + \hat{k} \cos\alpha \cos\omega$$

With formulas (34), we obtain, in the

K, ν - system:

$$\hat{i}x = + ix \cos K \cos\nu + jx \sin K \cos\nu - kx \sin\nu$$
$$\hat{j}y = - iy \sin K + jy \cos K \quad 0$$
$$\hat{k}c = + ic \cos K \sin\nu + jc \sin K \sin\nu + kc \cos\nu$$

α, ω - system:

$$\hat{i}x = + ix \cos\alpha \quad 0 \quad - kx \sin\alpha$$
$$\hat{j}y = - iy \sin\alpha \sin\omega + jy \cos\omega - ky \cos\alpha \sin\omega$$
$$\hat{k}c = + ic \sin\alpha \cos\omega + jc \sin\omega + kc \cos\alpha \cos\omega$$

Consequently, we have

$$\vec{r} = i(x \cos K \cos \nu - y \sin K + c \cos K \sin \nu)$$
$$+ j(x \sin K \cos \nu + y \cos K + c \sin K \sin \nu + k(-x \sin \nu + c \cos \nu) \tag{38}$$

or

$$\vec{r} = i(x \cos \alpha - y \sin \alpha \sin \omega + c \sin \alpha \cos \omega) + j(y \cos \omega + c \sin \omega)$$
$$+ k(-x \sin \alpha - y \cos \alpha \sin \omega + c \cos \alpha \cos \omega)$$

Again from formula (34)

for the K, ν - system:

$$iu = +\hat{i}u \cos K \cos \nu - \hat{j}u \sin K + \hat{k}u \cos K \sin \nu$$
$$jv = +\hat{i}v \sin K \cos \nu + \hat{j}v \cos K + \hat{k}v \sin K \sin \nu$$
$$kw = -\hat{i}w \sin \nu \qquad 0 \qquad + \hat{k}w \cos \nu$$

and for the α, ω - system:

$$iu = +\hat{i}u \cos \alpha - \hat{j}u \sin \alpha \sin \omega + \hat{k}u \sin \alpha \cos \omega$$
$$jv = \qquad 0 \qquad + \hat{j}v \cos \omega \qquad + \hat{k}v \sin \omega$$
$$kw = -\hat{i}w \sin \alpha - \hat{j}w \cos \alpha \sin \omega + \hat{k}w \cos \alpha \cos \omega$$

or

$$\vec{r} = \hat{i}(u \cos K \cos \nu + v \sin K \cos \nu - w \sin \nu) + \hat{j}(-u \sin K + v \cos K)$$
$$+ \hat{k}(u \cos K \sin \nu + v \sin K \sin \nu + w \cos \nu)$$

$$(39)$$

or

$$\vec{r} = \hat{i}(u \cos \alpha - w \sin \alpha) + \hat{j}(-u \sin \alpha \sin \omega + v \cos \omega - w \cos \alpha \sin \omega)$$
$$+ \hat{k}(u \sin \alpha \cos \omega + v \sin \omega + w \cos \alpha \cos \omega)$$

Consequently, again with formulas (34) from formulas (38) and (39) in the K, ν - system:

$$u = x \cos K \cos \nu - y \sin K + c \cos K \sin \nu$$
$$v = x \sin K \cos \nu + y \cos K + c \sin K \sin \nu \tag{40}$$
$$w = -x \sin \nu + c \cos \nu$$

α, ω - system:

$$u = x\cos\alpha - y\sin\alpha\sin\omega + c\sin\alpha\cos\omega$$

$$v = y\cos\omega + c\sin\omega \qquad (40)\text{ cont.}$$

$$w = -x\sin\alpha - y\cos\alpha\sin\omega + c\cos\alpha\cos\omega$$

or correspondingly in the K, ν - system:

$$x = +u\cos K\cos\nu + v\sin K\cos\nu - w\sin\nu$$

$$y = -u\sin K + v\cos K$$

$$c = +u\cos K\sin\nu + v\sin K\sin\nu + w\cos\nu$$

(41)

and in the α, ω - system:

$$x = +u\cos\alpha - w\sin\alpha$$

$$y = -u\sin\alpha\sin\omega + v\cos\omega - w\cos\alpha\sin\omega$$

$$c = +u\sin\alpha\cos\omega + v\sin\omega + w\cos\alpha\cos\omega$$

The coordinates (X) and (Y) for any point along the vector $\vec{R}$ for a certain elevation (Z) are:

$$(X) = \frac{u}{w}(Z) \qquad (Y) = \frac{v}{w}(Z)$$

With formulas (40) we obtain:

$$(X) = \frac{(Z)\ (x\cos K\cos\nu - y\sin K + c\cos K\sin\nu)}{-x\sin\nu + c\cos\nu}$$

$$= \frac{(Z)\ (x\cos\alpha - y\sin\alpha\sin\omega + c\sin\alpha\cos\omega)}{-x\sin\alpha - y\cos\alpha\sin\omega + c\cos\alpha\cos\omega}$$

$$(Y) = \frac{(Z)\ (x\sin K\cos\nu + y\cos K + c\sin K\sin\nu)}{-x\sin\nu + c\cos\nu} \qquad (42)$$

$$= \frac{(Z)\ (y\cos\omega + c\sin\omega)}{-x\sin\alpha - y\cos\alpha\sin\omega + c\cos\alpha\cos\omega}$$

Further, the object vector $\vec{R}$ may be expressed by

$\vec{R} = \mu\vec{r}$ and consequently $(X) = \mu u$

$(Y) = \mu v$ (43)

$(Z) = \mu w$ where μ is a scale factor

From Figure 1 we obtain $\vec{r} \cdot \hat{k} = c$, and with $\vec{R} \cdot \hat{k} = \mu \cdot \vec{r} \cdot \hat{k}$ and consequently $\mu = \frac{\vec{R}\,\hat{k}}{c}$, from formulas (34) and (37) we obtain

$$\mu = \frac{(X)\cos K \sin \nu + (Y)\sin K \sin \nu + (Z)\cos \nu}{c}$$

$$= \frac{(X)\sin \alpha \cos \omega + (Y)\sin \omega + (Z)\cos \alpha \cos \omega}{c}$$

and from formulas (41) and (43) we obtain for the K, ν - system:

$$x = \frac{c(+(X)\cos K \cos \nu + (Y)\sin K \cos \nu - (Z)\sin \nu)}{(X)\cos K \sin \nu + (Y)\sin K \sin \nu + (Z)\cos \nu} \tag{44a}$$

$$y = \frac{c(-(X)\sin K + (Y)\cos K)}{(X)\cos K \sin \nu + (Y)\sin K \sin \nu + (Z)\cos \nu}$$

and for the α, ω - system:

$$x = \frac{c(+(X)\cos \alpha - (Z)\sin \alpha)}{(X)\sin \alpha \cos \omega + (Y)\sin \omega + (Z)\cos \alpha \cos \omega} \tag{44b}$$

$$y = \frac{c(-(X)\sin \alpha \sin \omega + (Y)\cos \omega - (Z)\cos \alpha \sin \omega)}{(X)\sin \alpha \cos \omega + (Y)\sin \omega + (Z)\cos \alpha \cos \omega}$$

Further, from Figure 1:

$$(X) = X - X_o$$

$$(Y) = Y - Y_o$$

$$(Z) = Z - Z_o$$

for the K, ν - system:

$$x = -(\bar{x} - \bar{x}_p)\cos(\kappa) - (\bar{y} - \bar{y}_p)\sin(\kappa)$$

$$y = -(\bar{x} - \bar{x}_p)\sin(\kappa) + (\bar{y} - \bar{y}_p)\cos(\kappa)$$

for the α, ω - system: (44c)

$$x = -(\bar{x} - \bar{x}_p)\cos\kappa - (\bar{y} - \bar{y}_p)\sin\kappa$$

$$y = -(\bar{x} - \bar{x}_p)\sin\kappa + (\bar{y} - \bar{y}_p)\cos\kappa$$

From formulas (42) and (44) we obtain the final expressions:

$$X = \frac{(Z-Z_o)\left(\left\{c \sin\nu - \left[(\bar{x}-\bar{x}_p)\cos(\kappa) + (\bar{y}-\bar{y}_p)\sin(\kappa)\right]\cos\nu\right\}\cos K + \left[(\bar{x}-\bar{x}_p)\sin(\kappa) - (\bar{y}-\bar{y}_p)\cos(\kappa)\right]\sin K\right)}{c\cos\nu + \left[(\bar{x}-\bar{x}_p)\cos(\kappa) + (\bar{y}-\bar{y}_p)\sin(\kappa)\right]\sin\nu} + X_o \qquad (45a)^6$$

$$Y = \frac{(Z-Z_o)\left(\left\{c \sin\nu - \left[(\bar{x}-\bar{x}_p)\cos(\kappa) + (\bar{y}-\bar{y}_p)\sin(\kappa)\right]\cos\nu\right\}\sin K - \left[(\bar{x}-\bar{x}_p)\sin(\kappa) - (\bar{y}-\bar{y}_p)\cos(\kappa)\right]\cos K\right)}{c\cos\nu + \left[(\bar{x}-\bar{x}_p)\cos(\kappa) + (\bar{y}-\bar{y}_p)\sin(\kappa)\right]\sin\nu} + Y_o \qquad (45b)^6$$

$$X = \frac{(Z-Z_o)\left(\left\{c\cos\omega + \left[(\bar{x}-\bar{x}_p)\sin\kappa - (\bar{y}-\bar{y}_p)\cos\kappa\right]\sin\omega\right\}\sin\alpha - \left[(\bar{x}-\bar{x}_p)\cos\kappa + (\bar{y}-\bar{y}_p)\sin\kappa\right]\cos\alpha\right)}{\left\{c\cos\omega + \left[(\bar{x}-\bar{x}_p)\sin\kappa - (\bar{y}-\bar{y}_p)\cos\kappa\right]\sin\omega\right\}\cos\alpha + \left[(\bar{x}-\bar{x}_p)\cos\kappa + (\bar{y}-\bar{y}_p)\sin\kappa\right]\sin\alpha} + X_o \qquad (46a)^6$$

$$Y = \frac{(Z-Z_o)\left\{c\sin\omega - \left[(\bar{x}-\bar{x}_p)\sin\kappa - (\bar{y}-\bar{y}_p)\cos\kappa\right]\cos\omega\right\}}{\left\{c\cos\omega + \left[(\bar{x}-\bar{x}_p)\sin\kappa - (\bar{y}-\bar{y}_p)\cos\kappa\right]\sin\omega\right\}\cos\alpha + \left[(\bar{x}-\bar{x}_p)\cos\kappa + (\bar{y}-\bar{y}_p)\sin\kappa\right]\sin\alpha} + Y_o \qquad (46b)^6$$

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6. Formulas (45a) - (46b) are in agreement with the corresponding formulas (12a) - (13b) as given by v. Gruber in (5).

and

$$\bar{x} = \frac{c\left[(X-X_o)(-\cos K \cos \nu \cos(\kappa) + \sin K \sin(\kappa)) - (Y-Y_o)(\sin K \cos \nu \cos(\kappa) + \cos K \sin(\kappa)) + (Z-Z_o)(\sin \nu \cos(\kappa))\right]}{(X-X_o)\cos K \sin \nu + (Y-Y_o)\sin K \sin \nu + (Z-Z_o)\cos \nu} + \bar{x}_p \tag{47a}$$

$$\bar{y} = \frac{c\left[(X-X_o)(-\sin K \cos(\kappa) - \cos K \cos \nu \sin(\kappa)) + (Y-Y_o)(\cos K \cos(\kappa) - \sin K \cos \nu \sin(\kappa)) + (Z-Z_o)(\sin \nu \sin(\kappa))\right]}{(X-X_o)\cos K \sin \nu + (Y-Y_o)\sin K \sin \nu + (Z-Z_o)\cos \nu} + \bar{y}_p \tag{47b}$$

$$\bar{x} = \frac{c\left[(X-X_o)(-\cos \alpha \cos \kappa + \sin \alpha \sin \omega \sin \kappa) - (Y-Y_o)(\cos \omega \sin \kappa) + (Z-Z_o)(\sin \alpha \cos \kappa + \cos \alpha \sin \omega \sin \kappa)\right]}{(X-X_o)\sin \alpha \cos \omega + (Y-Y_o)\sin \omega + (Z-Z_o)\cos \alpha \cos \omega} + \bar{x}_p \tag{48a}$$

$$\bar{y} = \frac{c\left[(X-X_o)(-\sin \alpha \sin \omega \cos \kappa - \cos \alpha \sin \kappa) + (Y-Y_o)(\cos \omega \cos \kappa) - (Z-Z_o)(\cos \alpha \sin \omega \cos \kappa - \sin \alpha \sin \kappa)\right]}{(X-X_o)\sin \alpha \cos \omega + (Y-Y_o)\sin \omega + (Z-Z_o)\cos \alpha \cos \omega} + \bar{y}_p \tag{48b}$$

THE RIGOROUS LEAST SQUARES ADJUSTMENT

The numerical solution of the orientation problem and, consequently, the least squares adjustment for an overdetermined solution may be based either on the formulas (1) or (2), (45) or (47) and (46) or (48). In order to derive the advantage offered by formulas which are most useful with respect to the geometrical configuration, we will only consider the α, ω, κ-system, thus eliminating formulas (45) and (47) from further consideration. Furthermore, we must consider the fact that the presence of more than one observation and, therefore, more than one residual in a single observation equation calls for the introduction of additional weighting factors, a fact which complicates the numerical work considerably. Consequently, formulas (1) and (46) are less suitable for a rigorous least squares adjustment when the plate coordinates $\bar{x}$ and $\bar{y}$ are measured. Hence, formulas (2) and (48) seem to be the most suited expressions for a least squares treatment of the orientation problem. There is no special need for a numerical solution based on formulas (2) which express the mutual situation of two planes only and, thus, are but a special case of the general solution which is expressed by formulas (48). However, for the sake of comparison of the numerical work, both solutions are derived in the following chapters.

(a) Least Squares adjustment based on coefficients as given in formulas (2).

The observations of the plate coordinates with reference to a fiducial mark-system are denoted by ℓ and ℓ', corresponding to the $\bar{x}$ and $\bar{y}$ axes, respectively. The observational (residual) errors of these observations are denoted by v and v': hence,

$$\begin{aligned} \bar{x} &= \ell + v \\ \bar{y} &= \ell' + v' \end{aligned} \qquad (49)$$

The observation equations are, therefore:

$$\begin{aligned} v &= \frac{a_1'X + b_1'Y + c_1'}{a_o'X + b_o'Y + 1} - \ell \\ v' &= \frac{a_2'X + b_2'Y + c_2'}{a_o'X + b_o'Y + 1} - \ell' \end{aligned} \qquad (50)$$

With approximate values* for the coefficients, we have:

$$\begin{array}{lll} a_1' = a_1'^{o} + \Delta a_1' & a_2' = a_2'^{o} + \Delta a_2' & a_o' = a_o'^{o} + \Delta a_o' \\ b_1' = b_1'^{o} + \Delta b_1' & b_2' = b_2'^{o} + \Delta b_2' & b_o' = b_o'^{o} + \Delta b_o' \\ c_1' = c_1'^{o} + \Delta c_1' & c_2' = c_2'^{o} + \Delta c_2' & \end{array} \qquad (51)$$

* Approximate values are always denoted by $(^{o})$.

and the observation equations are:

$$v(1 + a_o'^{o}X + b_o'^{o}Y) = X\Delta a_1' + Y\Delta b_1' + \Delta c_1' - \ell X\Delta a_o' - \ell Y\Delta b_o' - \Delta\ell$$

$$v'(1 + a_o'^{o}X + b_o'^{o}Y) = X\Delta a_2' + Y\Delta b_2' + \Delta c_2' - \ell' X\Delta a_o' - \ell' Y\Delta b_o' - \Delta\ell'$$

where

$$\begin{aligned} -\Delta\ell &= a_1'^{o}X + b_1'^{o}Y + c_1'^{o} - \ell(1 + a_o'^{o}X + b_o'^{o}Y) \\ -\Delta\ell' &= a_2'^{o}X + b_2'^{o}Y + c_2'^{o} - \ell'(1 + a_o'^{o}X + b_o'^{o}Y) \end{aligned} \tag{52}$$

Both observation equations have the weight

$$p = \frac{1}{(1 + a_o'^{o}X + b_o'^{o}Y)^2}$$

These observation equations lead directly to the corresponding normal equations. However, to reduce the numerical computations it is advisable to eliminate $\Delta c_1'$ and $\Delta c_2'$ by forming reduced observation equations.

We have:

$$\begin{aligned} \Delta c_1' &= -\frac{[X]}{n}\Delta a_1' - \frac{[Y]}{n}\Delta b_1' + \frac{[\ell X]}{n}\Delta a_o' + \frac{[\ell Y]}{n}\Delta b_o' + \frac{[\Delta\ell]}{n} \\ \Delta c_2' &= -\frac{[X]}{n}\Delta a_2' - \frac{[Y]}{n}\Delta b_2' + \frac{[\ell' X]}{n}\Delta a_o' + \frac{[\ell' Y]}{n}\Delta b_o' + \frac{[\Delta\ell']}{n} \end{aligned} \tag{53}$$

and the reduced observation equations are:

$$\begin{aligned} \rho &= A\Delta a_1' + B\Delta b_1' + C\Delta a_o' + D\Delta b_o' - L \\ &\qquad\qquad \text{with the weight } p \\ \rho' &= A\Delta a_2' + B\Delta b_2' + C'\Delta a_o' + D'\Delta b_o' - L' \end{aligned} \tag{54}$$

$$p = \frac{1}{(1 + a_o'^{o}X + b_o'^{o}Y)^2}$$

where

$$A = X - \frac{[X]}{n}$$

$$B = Y - \frac{[Y]}{n}$$

$$C = -\ell X + \frac{[\ell X]}{n} \qquad C' = -\ell' X + \frac{[\ell' X]}{n} \tag{55}$$

$$D = -\ell Y + \frac{[\ell Y]}{n} \qquad D' = -\ell' Y + \frac{[\ell' Y]}{n}$$

$$-L = -\Delta\ell + \frac{[\Delta\ell]}{n} \qquad -L' = -\Delta\ell' + \frac{[\Delta\ell']}{n}$$

$$v = \rho\sqrt{p} \qquad v' = \rho'\sqrt{p'} .$$

These equations lead finally to the system of combined normal equations:

$$\begin{array}{cccccccc}
\Delta a_1' & \Delta b_1' & \Delta a_2' & \Delta b_2' & \Delta a_0' & \Delta b_0' & L & \\
\underline{[pAA]} & +[pAB] & 0 & 0 & +[pAC] & +[pAD] & -[pAL] & = 0 \\
 & \underline{[pBB]} & 0 & 0 & +[pBC] & +[pBD] & -[pBL] & = 0 \\
 & & \underline{[pAA]} & +[pAB] & +[pAC'] & +[pAD'] & -[pAL'] & = 0 \\
 & & & \underline{[pBB]} & +[pBC'] & +[pBD'] & -[pBL'] & = 0 \\
 & & & & \underline{[pCC + pC'C']} & +[pCD + pC'D'] & -[pCL + pC'L'] & = 0 \\
 & & & & & \underline{[pDD + pD'D']} & -[pDL + pD'L'] & = 0 \\
 & & & & & & \underline{[pLL + pL'L']} & = 0
\end{array}$$

After the unknowns $\Delta a_1'$, $\Delta b_1'$, $\Delta a_2'$, $\Delta b_2'$, $\Delta a_0'$ and $\Delta b_0'$ are determined, $\Delta c_1'$ and $\Delta c_2'$ are obtained from formulas (53), the residuals ρ and ρ' from formulas (54) and v and v' from formulas (55). The final coefficients are computed from formulas (51). During the computations we have the following checks: $[pA\rho] + [pA\rho'] = 0$, $[pB\rho] + [pB\rho'] = 0$, etc. and $[vv] + [v'v'] = [p\rho\rho] + [p\rho'\rho'] = [pLL.6]$. The final check is made by means of formulas (2). The computed x and y values must be in complete agreement with the adjusted observations $(\ell + v)$ and $(\ell' + v')$. If the Δ - values are large, iteration becomes necessary due to the neglected second order terms.

The mean error of unit weight of an observed plate coordinate is

$$m = \pm\sqrt{\frac{[vv] + [v'v']}{2n - 8}} \tag{56}$$

The mean errors of the coefficients are then computed by multiplying m by the square root of the corresponding weight coefficient which may be obtained, during the reduction of the normal equation - system, by applying the indirect solution using, e.g. the method of Cholesky, E. Anderson or H. Wolf. If later the elements of orientation are computed from the final coefficients with formulas (6) - (11) and (19) - (25) and the corresponding mean errors are required, it is necessary to compute each of these as a mean error of a function F of the unknowns, or even as a mean error of a function of functions of the unknowns. The necessary weighting factors may in such a case also be obtained during the reduction process of the normal equations by carrying additional dF - columns. If the elements of interior orientation are given, as generally in aerial photogrammetry, or if the center of projection is known, as is in general the case in ground photogrammetry, there are only six degrees of freedom present and, consequently, in addition to the formulas (50), the existing condition equations (27) must be introduced in the least squares adjustment. By means of the approximate values and the Taylor series, neglecting terms of second and higher order, we obtain from formulas (27):

$$a_2'^{o} \Delta a_1' + a_1'^{o} \Delta a_2' + b_2'^{o} \Delta b_1' + b_1'^{o} \Delta b_2' + c_2'^{o} \Delta c_1' + c_1'^{o} \Delta c_2' + \lambda_1 = 0$$

where $\lambda_1 = a_1'^{o} a_2'^{o} + b_1'^{o} b_2'^{o} + c_1'^{o} c_2'^{o}$

(57)

and $a_1'^{o} \Delta a_1' + b_1'^{o} \Delta b_1' + c_1'^{o} \Delta c_1' - a_2'^{o} \Delta a_2' - b_2'^{o} \Delta b_2' - c_2'^{o} \Delta c_2' + \lambda_2 = 0$

where $\lambda_2 = \frac{1}{2} (a_1'^{o^2} + b_1'^{o^2} + c_1'^{o^2} - a_2'^{o^2} - b_2'^{o^2} - c_2'^{o^2})$

For ease of computing, the approximate values of the coefficients should be chosen in such a way that they satisfy the condition equations (27), thus making both λ_1 and λ_2 equal to zero. This case only will be considered.

The following expressions are helpful for this step.

$$a_2' = \sqrt{\frac{1}{2}(c_1'^2 + b_1'^2 - c_2'^2 - b_2'^2) \pm \sqrt{\frac{1}{4}(c_1'^2 + b_1'^2 - c_2'^2 - b_2'^2)^2 + c_1'^2 c_2'^2 + b_1'^2 b_2'^2 + 2c_1 c_2 b_1 b_2}}$$

$$a_1' = \frac{-b_1' b_2' - c_1' c_2'}{a_2'}$$

After suitable transformations we obtain from formulas (57)

$$\Delta a_1' = a' \Delta b_1' + b' \Delta b_2' + c' \Delta c_1' + d' \Delta c_2'$$

(58)

$$\Delta a_2' = -b' \Delta b_1' + a' \Delta b_2' - d' \Delta c_1' + c' \Delta c_2'$$

where

$$a' = \frac{-(a_1'^{o} b_1'^{o} + a_2'^{o} b_2'^{o})}{a_1'^{o^2} + a_2'^{o^2}} \qquad c' = \frac{-(a_1'^{o} c_1'^{o} + a_2'^{o} c_2'^{o})}{a_1'^{o^2} + a_2'^{o^2}}$$

$$b' = \frac{a_1'^{o} b_2'^{o} - a_2'^{o} b_1'^{o}}{a_1'^{o^2} + a_2'^{o^2}} \qquad d' = \frac{a_1'^{o} c_2'^{o} - a_2'^{o} c_1'^{o}}{a_1'^{o^2} + a_2'^{o^2}} \tag{59}$$

Substituting formulas (58) in (52), the observation equations are:

$$v(1 + a_o'^{o}X + b_o'^{o}Y) = (Y + Xa')\Delta b_1' + Xb'\Delta b_2' + (1 + Xc')\Delta c_1' + Xd'\Delta c_2' - \ell X\Delta a_o'$$
$$- \ell Y\Delta b_o' - \Delta \ell$$

$$v'(1 + a_o'^{o}X + b_o'^{o}Y) = -Xb'\Delta b_1' + (Y + Xa')\Delta b_2' - Xd'\Delta c_1' + (1 + Xc')\Delta c_2' - \ell' X\Delta a_o'$$
$$- \ell' Y\Delta b_o' - \Delta \ell'$$

These observation equations may be written as

$$\rho = A\,\Delta b_1' + B\,\Delta b_2' + C\,\Delta c_1' + D\,\Delta c_2' + E\,\Delta a_o' + F\,\Delta b_o' - \Delta \ell$$
$$\rho' = -B\,\Delta b_1' + A\,\Delta b_2' - D\,\Delta c_1' + C\,\Delta c_2' + E'\,\Delta a_o' + F'\,\Delta b_o' - \Delta \ell' \tag{60}$$

where

$A = Y + Xa'$

$B = Xb'$

$C = 1 + Xc'$

$D = Xd'$

$E = -\ell X \qquad E' = -\ell' X$

$F = -\ell Y \qquad F' = -\ell' Y$

$-\Delta \ell = a_1'^{o}X + b_1'^{o}Y + c_1'^{o} - \ell\,(1 + a_o'^{o}X + b_o'^{o}Y)$

$-\Delta \ell' = a_2'^{o}X + b_2'^{o}Y + c_2'^{o} - \ell'(1 + a_o'^{o}X + b_o'^{o}Y)$

Both observation equations have the weight

$$p = \frac{1}{(1 + a_o'^o X + b_o'^o Y)^2}$$

and

$$\begin{aligned} v &= \rho \sqrt{p} \\ v' &= \rho' \sqrt{p} \end{aligned} \qquad (61)$$

The observation equations (60) lead directly to the normal equation system.

$$\begin{array}{ccccccc}
\Delta b_1' & \Delta b_2' & \Delta c_1' & \Delta c_2' & \Delta a_o' & \Delta b_o' & \Delta \ell \\
[pAA+pBB] & 0 & +[pAC+pBD] & +[pAD-pBC] & +[pAE-pBE'] & +[pAF-pBF'] & -[pA\Delta\ell-pB\Delta\ell'] = 0 \\
 & [pAA+pBB] & -[pAD-pBC] & +[pAC+pBD] & +[pBE+pAE'] & +[pBF+pAF'] & -[pB\Delta\ell+pA\Delta\ell'] = 0 \\
 & & [pCC+pDD] & 0 & +[pCE-pDE'] & +[pCF-pDF'] & -[pC\Delta\ell-pD\Delta\ell'] = 0 \\
 & & & [pCC+pDD] & +[pDE+pCE'] & +[pDF+pCF'] & -[pD\Delta\ell+pC\Delta\ell'] = 0 \\
 & & & & [pEE+pE'E'] & +[pEF+pE'F'] & -[pE\Delta\ell+pE'\Delta\ell'] = 0 \\
 & (62) & & & & [pFF+pF'F'] & -[pF\Delta\ell+pF'\Delta\ell'] = 0 \\
 & & & & & & [p\Delta\ell\Delta\ell+p\Delta\ell'\Delta\ell'] = 0
\end{array}$$

This system offers an advantage for numerical computations because only twenty-two of the twenty-eight coefficients must be computed. After the unknowns $\Delta b_1'$, $\Delta b_2'$, $\Delta c_1'$, $\Delta c_2'$, $\Delta a_o'$, $\Delta b_o'$ are determined, $\Delta a_1'$ and $\Delta a_2'$ are computed from formulas (58), and the final coefficients from formulas (51). As a first check the computed coefficients are introduced in the condition equations (27). In case the Δ-values are large, iteration becomes necessary before the condition equations are sufficiently satisfied, due to the neglected second order terms. The residuals ρ and ρ' are computed from formulas (60) and v and v' from (61). The adjusted observations $\bar{x}$ and $\bar{y}$ are obtained from formulas (49) and the final check is carried out by computing a set of $\bar{x}$ and $\bar{y}$ coordinates with the final coefficients and formulas (2). These values must agree completely with the adjusted observations $\bar{x} = (\ell + v)$ and $\bar{y} = (\ell' + v')$. During the computing a further check is made by means of

$$[vv] + [v'v'] = [p\rho\rho] + [p\rho'\rho'] = [p\Delta\ell\Delta\ell.6]$$

$$\text{and } [pA\rho] = [pB\rho'] = 0, \quad [pB\rho] + [pA\rho'] = 0, \text{ etc.}$$

The mean error of an observed coordinate of unit weight is

$$m = \pm\sqrt{\frac{[vv] + [v'v']}{2n - 6}} \tag{63}$$

The computation of the mean error of the coefficients must be carried out by the procedure given on page 34. The elements of orientation are computed from the final coefficients with formulas (19), (21), (23) - (25) or (28) - (33), respectively. The mean errors of these quantities must again be computed as mean errors of functions of the unknowns as explained above.

(b) Least Squares adjustment of the general solution based on formulas (48).

The observations of the plate coordinates with reference to an arbitrarily oriented fiducial mark - system are again denoted by ℓ and ℓ' corresponding to the $\bar{x}$ and $\bar{y}$ axes, respectively. The observational errors of these observations are v and v'. Hence we have again formulas (49)

$$\bar{x} = \ell + v$$

$$\bar{y} = \ell' + v'$$

From formulas (48) we obtain the observation equations

$$\ell + v = \frac{c\left[(X-X_o)A + (Y-Y_o)B + (Z-Z_o)C\right]}{(X-X_o)D + (Y-Y_o)E + (Z-Z_o)F} + \bar{x}_p = G$$

$$\ell' + v' = \frac{c\left[(X-X_o)A' + (Y-Y_o)B' + (Z-Z_o)C'\right]}{(X-X_o)D + (Y-Y_o)E + (Z-Z_o)F} + \bar{y}_p = G' \tag{64}$$

where

$A = -\cos\alpha\cos\kappa + \sin\alpha\sin\omega\sin\kappa \qquad A' = -\cos\alpha\sin\kappa - \sin\alpha\sin\omega\cos\kappa$

$B = -\cos\omega\sin\kappa \qquad B' = \cos\omega\cos\kappa$

$C = \sin\alpha\cos\kappa + \cos\alpha\sin\omega\sin\kappa \qquad C' = \sin\alpha\sin\kappa - \cos\alpha\sin\omega\cos\kappa$

$D = \sin\alpha\cos\omega$

$E = \sin\omega$

$F = \cos\alpha\cos\omega$

From the Taylor expansion for the right hand side of the above equations, we have, neglecting terms of second and higher order:

$$v = \frac{\partial G}{\partial \alpha}\Delta\alpha + \frac{\partial G}{\partial \omega}\Delta\omega + \frac{\partial G}{\partial \kappa}\Delta\kappa + \frac{\partial G}{\partial c}\Delta c + \frac{\partial G}{\partial X_o}\Delta X_o + \frac{\partial G}{\partial Y_o}\Delta Y_o + \frac{\partial G}{\partial Z_o}\Delta Z_o + \frac{\partial G}{\partial \bar{x}_p}\Delta \bar{x}_p - \Delta \ell \tag{65}$$

$$v' = \frac{\partial G'}{\partial \alpha}\Delta\alpha + \frac{\partial G'}{\partial \omega}\Delta\omega + \frac{\partial G'}{\partial \kappa}\Delta\kappa + \frac{\partial G'}{\partial c}\Delta c + \frac{\partial G'}{\partial X_o}\Delta X_o + \frac{\partial G'}{\partial Y_o}\Delta Y_o + \frac{\partial G'}{\partial Z_o}\Delta Z_o + \frac{\partial G'}{\partial \bar{y}_p}\Delta \bar{y}_p - \Delta \ell'$$

where $-\Delta\ell = \ell^o - \ell$ $\qquad -\Delta\ell' = \ell'^o - \ell'$

ℓ^o and ℓ'^o are computed with formulas (64) and the approximate values, for which

$$\begin{aligned} \alpha &= \alpha^o + \Delta\alpha & X_o &= X_o^o + \Delta X_o & c &= c^o + \Delta c \\ \omega &= \omega^o + \Delta\omega & Y_o &= Y_o^o + \Delta Y_o & \bar{x}_p &= \bar{x}_p^o + \Delta\bar{x}_p \\ \kappa &= \kappa^o + \Delta\kappa & Z_o &= Z_o^o + \Delta Z_o & \bar{y}_p &= \bar{y}_p^o + \Delta\bar{y}_p \end{aligned} \tag{66}$$

Making use of formulas (44) and (48), the observation equations (65) may be written, introducing the approximate values (66):

$$\begin{aligned} v = &\left[c^o B'^o - (\ell'^o - \bar{y}_p^o)E^o - \frac{\ell^o - \bar{x}_p^o}{c^o} \cos\omega^o\, x^o \right]\Delta\alpha \\ &+ \left(c^o \sin\kappa^o - \frac{\ell^o - \bar{x}_p^o}{c^o}\, y^o \right)\Delta\omega \\ &- (\ell'^o - \bar{y}_p^o)\,\Delta\kappa \\ &+ \frac{\ell^o - \bar{x}_p^o}{c^o}\,\Delta c \\ &+ \frac{-c^o A^o + (\ell^o - \bar{x}_p^o)D^o}{q}\,\Delta X_o \\ &+ \frac{-c^o B^o + (\ell^o - \bar{x}_p^o)E^o}{q}\,\Delta Y_o \\ &+ \frac{-c^o C^o + (\ell^o - \bar{x}_p^o)F^o}{q}\,\Delta Z_o \\ &+ \Delta\bar{x}_p \\ &- \Delta\ell \end{aligned} \tag{67}$$

$$\text{and} \quad v' = \left[-c^o B^o + (\ell^o - \bar{x}_p^o) E^o - \frac{\ell'^o - \bar{y}_p^o}{c^o} \cos \omega^o x^o \right] \Delta \alpha$$

$$+ \left(c^o \cos \kappa^o + \frac{\ell'^o - \bar{y}_p^o}{c^o} y^o \right) \Delta \omega$$

$$+ (\ell^o - \bar{x}_p^o) \Delta \kappa$$

$$+ \frac{\ell'^o - \bar{y}_p^o}{c^o} \Delta c \qquad (67) \text{ cont.}$$

$$+ \frac{-c^o A'^o + (\ell'^o - \bar{y}_p^o) D^o}{q} \Delta X_o$$

$$+ \frac{-c^o B'^o + (\ell'^o - \bar{y}_p^o) E^o}{q} \Delta Y_o$$

$$+ \frac{-c^o C'^o + (\ell'^o - \bar{y}_p^o) F^o}{q} \Delta Z_o$$

$$+ \Delta \bar{y}_p$$

$$- \Delta \ell'$$

where

$$x^o = -(\ell^o - \bar{x}_p^o) \cos \kappa^o - (\ell'^o - \bar{y}_p^o) \sin \kappa^o; \qquad -\Delta \ell = \ell^o - \ell$$

$$y^o = -(\ell^o - \bar{x}_p^o) \sin \kappa^o + (\ell'^o - \bar{y}_p^o) \cos \kappa^o; \qquad -\Delta \ell' = \ell'^o - \ell'$$

and

$$\ell^o = \frac{c^o \left[(X-X_o^o) A^o + (Y-Y_o^o) B^o + (Z-Z_o^o) C^o \right]}{(X-X_o^o) D^o + (Y-Y_o^o) E^o + (Z-Z_o^o) F^o} + \bar{x}_p^o = \frac{c^o m}{q} + \bar{x}_p^o$$

$$\ell'^o = \frac{c^o \left[(X-X_o^o) A'^o + (Y-Y_o^o) B'^o + (Z-Z_o^o) C'^o \right]}{(X-X_o^o) D^o + (Y-Y_o^o) E^o + (Z-Z_o^o) F^o} + \bar{y}_p^o = \frac{c^o n}{q} + \bar{y}_p^o$$

and

$$A^o = -\cos \alpha^o \cos \kappa^o + \sin \alpha^o \sin \omega^o \sin \kappa^o$$

$$A'^o = -\cos \alpha^o \sin \kappa^o - \sin \alpha^o \sin \omega^o \cos \kappa^o$$

$$B^o = -\cos \omega^o \sin \kappa^o$$

$$B'^o = +\cos \omega^o \cos \kappa^o$$

$$C^{o} = \sin \alpha^{o} \cos \kappa^{o} + \cos \alpha^{o} \sin \omega^{o} \sin \kappa^{o}$$

$$C'^{o} = \sin \alpha^{o} \sin \kappa^{o} - \cos \alpha^{o} \sin \omega^{o} \cos \kappa^{o}$$

$$D^{o} = \sin \alpha^{o} \cos \omega^{o}$$

$$E^{o} = \sin \omega^{o}$$

$$F^{o} = \cos \alpha^{o} \cos \omega^{o}$$

The observation equations may now be written

$$
\begin{aligned}
v &= a\,\Delta\alpha + b\,\Delta\omega + c\,\Delta\kappa + d\,\Delta c + e\,\Delta X_o + f\,\Delta Y_o + g\,\Delta Z_o + \Delta\bar{x}_p - \Delta\xi \\
v' &= a'\,\Delta\alpha + b'\,\Delta\omega + c'\,\Delta\kappa + d'\,\Delta c + e'\,\Delta X_o + f'\,\Delta Y_o + g'\,\Delta Z_o + \Delta\bar{y}_p - \Delta\xi'
\end{aligned}
\qquad (68)
$$

where the meaning of the coefficients a-g and a'-g' is given by the formulas (67). The simplicity and symmetry of these coefficients contribute considerably to the economy of the computations.

Before we form normal equations, we eliminate the unknowns $\Delta\bar{x}_p$ and $\Delta\bar{y}_p$.

$$
\begin{aligned}
\Delta\bar{x}_p &= -\frac{[a]}{n}\Delta\alpha - \frac{[b]}{n}\Delta\omega - \frac{[c]}{n}\Delta\kappa - \frac{[d]}{n}\Delta c - \frac{[e]}{n}\Delta X_o - \frac{[f]}{n}\Delta Y_o - \frac{[g]}{n}\Delta Z_o + \frac{[\Delta\xi]}{n} \\
\Delta\bar{y}_p &= -\frac{[a']}{n}\Delta\alpha - \frac{[b']}{n}\Delta\omega - \frac{[c']}{n}\Delta\kappa - \frac{[d']}{n}\Delta c - \frac{[e']}{n}\Delta X_o - \frac{[f']}{n}\Delta Y_o - \frac{[g']}{n}\Delta Z_o + \frac{[\Delta\xi']}{n}
\end{aligned}
\qquad (69)
$$

The reduced observation equations are now

$$
\begin{aligned}
v &= (a)\,\Delta\alpha + (b)\,\Delta\omega + (c)\,\Delta\kappa + (d)\,\Delta c + (e)\,\Delta X_o + (f)\,\Delta Y_o + (g)\,\Delta Z_o - L \\
v' &= (a)'\,\Delta\alpha + (b)'\,\Delta\omega + (c)'\,\Delta\kappa + (d)'\,\Delta c + (e)'\,\Delta X_o + (f)'\,\Delta Y_o + (g)'\,\Delta Z_o - L'
\end{aligned}
\qquad (70)
$$

where

$$(a) = a - \frac{[a]}{n} \qquad (a)' = a' - \frac{[a']}{n}$$

$$(b) = b - \frac{[b]}{n} \qquad (b)' = b' - \frac{[b']}{n}$$

$$(c) = c - \frac{[c]}{n} \qquad (c)' = c' - \frac{[c']}{n}$$

$$(d) = d - \frac{[d]}{n} \qquad (d)' = d' - \frac{[d']}{n}$$

$$(e) = e - \frac{[e]}{n} \qquad (e)' = e' - \frac{[e']}{n}$$

$$(f) = f - \frac{[f]}{n} \qquad (f)' = f' - \frac{[f']}{n}$$

$$(g) = g - \frac{[g]}{n} \qquad (g)' = g' - \frac{[g']}{n}$$

$$-L = -\Delta\xi + \frac{[\Delta\xi]}{n} \qquad -L' = -\Delta\xi' + \frac{[\Delta\xi']}{n}$$

We now obtain directly the corresponding combined normal equations:

$$
\begin{array}{ccccccccccccccccc}
\Delta\alpha & & \Delta\omega & & \Delta\ell & & \Delta c & & \Delta X_o & & \Delta Y_o & & \Delta Z_o & & L & & \\
[(a)(a)+(a)'(a)'] & + & [(a)(b)+(a)'(b)'] & + & [(a)(c)+(a)'(c)'] & + & [(a)(d)+(a)'(d)'] & + & [(a)(e)+(a)'(e)'] & + & [(a)(f)+(a)'(f)'] & + & [(a)(g)+(a)'(g)'] & - & [(a)L+(a)'L'] & = & 0 \\
 & & [(b)(b)+(b)'(b)'] & + & [(b)(c)+(b)'(c)'] & + & [(b)(d)+(b)'(d)'] & + & [(b)(e)+(b)'(e)'] & + & [(b)(f)+(b)'(f)'] & + & [(b)(g)+(b)'(g)'] & - & [(b)L+(b)'L'] & = & 0 \\
 & & & & [(c)(c)+(c)'(c)'] & + & [(c)(d)+(c)'(d)'] & + & [(c)(e)+(c)'(e)'] & + & [(c)(f)+(c)'(f)'] & + & [(c)(g)+(c)'(g)'] & - & [(c)L+(c)'L'] & = & 0 \\
 & & & & & & [(d)(d)+(d)'(d)'] & + & [(d)(e)+(d)'(e)'] & + & [(d)(f)+(d)'(f)'] & + & [(d)(g)+(d)'(g)'] & - & [(d)L+(d)'L'] & = & 0 \\
 & & & & & & & & [(e)(e)+(e)'(e)'] & + & [(e)(f)+(e)'(f)'] & + & [(e)(g)+(e)'(g)'] & - & [(e)L+(e)'L'] & = & 0 \\
 & & & & & & & & & & [(f)(f)+(f)'(f)'] & + & [(f)(g)+(f)'(g)'] & - & [(f)L+(f)'L'] & = & 0 \\
 & & & & & & & & & & & & [(g)(g)+(g)'(g)'] & - & [(g)L+(g)'L'] & = & 0 \\
 & & & & & & & & & & & & & & [LL+L'L'] & = & 0
\end{array}
\tag{71}
$$

After the unknowns $\Delta\alpha$, $\Delta\omega$, $\Delta\kappa$, Δc, ΔX_o, ΔY_o, ΔZ_o are computed, we obtain $\Delta\bar{x}_p$ and $\Delta\bar{y}_p$ from formula (69). The residuals v and v' are computed from formulas (70). The final orientation elements are obtained from formulas (66) and the adjusted observations from formulas (49). During the computations we have the usual checks: $[(a)v] + [(a)'v'] = 0$, $[(b)v] + [(b)'v'] = 0$, etc., and $[vv] + [v'v'] = [LL.7]$. The final check is obtained with the final orientation elements and formulas (64) which must give for G and G' the adjusted observations $(\ell + v)$ and $(\ell' + v')$. If the corrections Δ are large, iteration becomes necessary due to the neglected second order terms. The mean error of an observed coordinate is:

$$m = \pm\sqrt{\frac{[vv] + [v'v']}{2n - N}} \tag{72}$$

where N is the number of unknowns carried. The mean errors of the elements of orientation are obtained by multiplying m by the square root of the corresponding weighting factors. These are directly obtained from the normal equations if for their reduction the method of the indirect solution is applied.

In general, the electronic computing devices are very useful for any solution based on iteration. Consequently, such equipment seems especially suitable for the above derived analytical solution of the orientation problem. On the other hand, electronic computers are less suited to handle the transformation of angles with their trigonometric functions and vice versa. This difficulty can be solved for our problem with the following substitutions. In the formulas (67) we introduce the auxiliary unknowns:

$$\begin{aligned} r &= \sin\alpha \quad \text{and consequently} \quad & r &= r^o + \Delta r \quad \text{where} \quad & r^o &= \sin\alpha^o \\ s &= \sin\omega & s &= s^o + \Delta s & s^o &= \sin\omega^o \\ t &= \sin\kappa & t &= t^o + \Delta t & t^o &= \sin\kappa^o \end{aligned} \tag{73}$$

Furthermore

$$r'^o = \cos\alpha^o = \sqrt{1 - r^{o2}}\ ; \quad s'^o = \cos\omega^o = \sqrt{1 - s^{o2}} \quad \text{and} \quad t'^o = \cos\kappa^o = \sqrt{1 - t^{o2}}$$

We have then

$$\begin{aligned} A^o &= -r'^o t'^o + r^o s^o t^o & A'^o &= -r'^o t^o - r^o s^o t'^o & D^o &= r^o s'^o \\ B^o &= -s'^o t^o & B'^o &= s'^o t'^o & E^o &= s^o \\ C^o &= r^o t'^o + r'^o s^o t^o & C'^o &= r^o t^o - r'^o s^o t'^o & F^o &= r'^o s'^o \end{aligned}$$

and the observation equations are now:

$$
\begin{aligned}
v = \; & \left[c^o B'^o - (\ell'^o - \bar{y}^o_p) E^o - \frac{\ell^o - \bar{x}^o_p}{c^o} \, s'^o x^o \right] \Delta \alpha \\
& + \left(c^o t^o - \frac{\ell^o - \bar{x}^o_p}{c^o} \, y^o \right) \Delta \omega \\
& - (\ell'^o - \bar{y}^o_p) \Delta \kappa \\
& + \frac{\ell^o - \bar{x}^o_p}{c^o} \, \Delta c \\
& + \frac{-c^o A^o + (\ell^o - \bar{x}^o_p) D^o}{q} \, \Delta X_o \\
& + \frac{-c^o B^o + (\ell^o - \bar{x}^o_p) E^o}{q} \, \Delta Y_o \\
& + \frac{-c^o C^o + (\ell^o - \bar{x}^o_p) F^o}{q} \, \Delta Z_o \\
& + \Delta \bar{x}_p \\
& - \Delta \ell
\end{aligned}
\qquad (74)
$$

$$
\begin{aligned}
v' = \; & \left[-c^o B^o + (\ell^o - \bar{x}^o_p) E^o - \frac{\ell'^o - \bar{y}^o_p}{c^o} \, s'^o x^o \right] \Delta \alpha \\
& - \left(c^o t'^o + \frac{\ell'^o - \bar{y}^o_p}{c^o} \, y^o \right) \Delta \omega \\
& + (\ell^o - \bar{x}^o_p) \Delta \kappa \\
& + \frac{\ell'^o - \bar{y}^o_p}{c^o} \, \Delta c \\
& + \frac{-c^o A'^o + (\ell'^o - \bar{y}^o_p) D^o}{q} \, \Delta X_o \\
& + \frac{-c^o B'^o + (\ell'^o - \bar{y}^o_p) E^o}{q} \, \Delta Y_o \\
& + \frac{-c^o C'^o + (\ell'^o - \bar{y}^o_p) F^o}{q} \, \Delta Z_o \\
& + \Delta \bar{y}_p \\
& - \Delta \ell'
\end{aligned}
$$

where

$$x^{o} = -(\ell^{o} - \bar{x}_{p}^{o})t'^{o} - (\ell'^{o} - \bar{y}_{p}^{o})t^{o} \; ; \qquad -\Delta\ell = \ell^{o} - \ell$$

$$y^{o} = -(\ell^{o} - \bar{x}_{p}^{o})t^{o} + (\ell'^{o} - \bar{y}_{p}^{o})t'^{o} \; ; \qquad -\Delta\ell' = \ell'^{o} - \ell'$$

Furthermore

$$\begin{array}{lcl} r = \Delta\alpha\, r'^{o} & & \Delta r' = -\Delta\alpha\, r^{o} \\ s = \Delta\omega\, s'^{o} & \text{or} & \Delta s' = -\Delta\omega\, s^{o} \\ t = \Delta\kappa\, t'^{o} & & \Delta t' = -\Delta\kappa\, t^{o} \end{array} \tag{75}$$

From here the solution is based on iterations which are continued until a pre-established discrimination constant is obtained. Such a value may be established by means of a lower limit for the expression $\dfrac{|[\Delta\ell]| + |[\Delta\ell']|}{2n}$.

THE PYRAMID METHOD

The determination of the spatial coordinates of the nodal point and the orientation of a single photograph taken from an airborne camera is known as the problem of resection in space. In general, the elements of interior orientation are known. Thus the problem is a special case of the general solution which is explained in the preceding paragraphs. (Compare numerical example No. 3). However, several authors have spent a considerable amount of effort on theoretical and numerical studies concerned with an approach based on determining first the spatial location of the nodal point, with the help of a pyramid formed by the rays producing the photograph. The rotational components are then computed as a second step. The common opinion resulting from these studies is that the pyramid method is not useful if there are observations in excess, making necessary a least squares treatment. This conclusion is based on the assumption that each two of the given control points must be combined in an observation equation[7], thus leading, for n-points, to $\frac{n(n-1)}{2}$ observation equations. It will be shown that this assumption is incorrect

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[7]See, e.g., (6) page 8 or (10) page 66.

and that only (2n-3) independent observation equations exist. Hence, a least squares solution based on the pyramid method may well be advantageous in cases where only the location of the nodal point of the survey camera has to be determined.

The unique solution requires computing the lengths of the sides of a pyramid with a triangular base from the given lengths of the sides of the base triangle and the corresponding position angles at the vertex of the pyramid. The treatment of this problem leads to an equation of the fourth degree. Generally, it is possible to obtain approximate values for the position directly from the photograph, or, in case of strip photographs, by linear extrapolation of two preceding nodal point positions. In such a case, the problem is to compute corrections ΔX_o, ΔY_o and ΔZ_o to the approximate values X_o^o, Y_o^o and Z_o^o. Hugershoff[8] proposes the following method where these corrections are found directly from the position angles at the vertex of the pyramid.

The camera has the elements of interior orientation c, $\bar{x}_p$ and $\bar{y}_p$. Given are the spatial Cartesian coordinates X_i, Y_i, Z_i of the suitably located reference points, the corresponding plate measurements ℓ_i and ℓ_i', and the approximations X_o^o, Y_o^o and Z_o^o for the position of the nodal point.

We set:

$$
\begin{aligned}
X_o &= X_o^o + \Delta X_o \\
Y_o &= Y_o^o + \Delta Y_o \\
Z_o &= Z_o^o + \Delta Z_o
\end{aligned}
\qquad (76)
$$

Further, denoting the lengths of the sides of the pyramid above the photograph by ℓ and the corresponding lengths of the sides of the pyramid above the reference points by L, we have:

$$\bar{L}_i^2 = (X_o^o + \Delta X_o - X_i)^2 + (Y_o^o + \Delta Y_o - Y_i)^2 + (Z_o^o + \Delta Z_o - Z_i)^2 \qquad (77)$$

and

$$\bar{L}_i^2 + \bar{L}_j^2 - 2\bar{L}_i\bar{L}_j \cos \sigma_{ij} = \bar{D}_{ij}^2 \qquad (78)$$

where σ_{ij} is the spatial angle at the vertex of the pyramid and $\bar{D}_{ij}$ the slant ground distance between the reference points i, j. $\bar{D}_{ij}^2$ is obtained

---

[8] See (9)

from the given coordinates of the reference points by

$$\bar{D}_{ij}^2 = (X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2$$

Cos σ_{ij} is computed from the measured plate coordinates and the elements of interior orientation. Denoting the plate distance between the points i, j by $\bar{d}_{ij}$, we have:

$$\cos \sigma_{ij} = \frac{\bar{\ell}_i^2 + \bar{\ell}_j^2 - \bar{d}_{ij}^2}{2\bar{\ell}_i \bar{\ell}_j} \qquad \text{where } \bar{\ell}_i^2 = c^2 + (\ell_i - \bar{x}_p)^2 + (\ell_i' - \bar{y}_p)^2$$

and $\bar{d}_{ij}^2 = (\ell_j - \ell_i)^2 + (\ell_j' - \ell_i')^2$

Applying the Taylor series and neglecting second and higher order terms we obtain from formulas (77):

$$\bar{L}_i^2 = \bar{L}_i^{o^2} + 2(X_o^o - X_i)\Delta X_o + 2(Y_o^o - Y_i)\Delta Y_o + 2(Z_o^o - Z_i)\Delta Z_o$$

$$\bar{L}_i = \bar{L}_i^o + \frac{X_o^o - X_i}{\bar{L}^o}\Delta X_o + \frac{Y_o^o - Y_i}{\bar{L}^o}\Delta Y_o + \frac{Z_o^o - Z_i}{\bar{L}^o}\Delta Z_o \tag{79}$$

where $\bar{L}_i^{o^2} = (X_o^o - X_i)^2 + (Y_o^o - Y_i)^2 + (Z_o^o - Z_i)^2$

Inserting formulas (79) into formulas (78) we obtain, again neglecting second order terms,

$$\begin{aligned}
&+ \left[(X_o^o - X_i)\left(1 - \frac{\bar{L}_j^o \cos \sigma_{ij}}{\bar{L}_i^o}\right) + (X_o^o - X_j)\left(1 - \frac{\bar{L}_i^o \cos \sigma_{ij}}{\bar{L}_j^o}\right)\right]\Delta X_o \\
&+ \left[(Y_o^o - Y_i)\left(1 - \frac{\bar{L}_j^o \cos \sigma_{ij}}{\bar{L}_i^o}\right) + (Y_o^o - Y_j)\left(1 - \frac{\bar{L}_i^o \cos \sigma_{ij}}{\bar{L}_j^o}\right)\right]\Delta Y_o \\
&+ \left[(Z_o^o - Z_i)\left(1 - \frac{\bar{L}_j^o \cos \sigma_{ij}}{\bar{L}_i^o}\right) + (Z_o^o - Z_j)\left(1 - \frac{\bar{L}_i^o \cos \sigma_{ij}}{\bar{L}_j^o}\right)\right]\Delta Z_o \\
&+ \frac{\bar{L}_i^{o^2} + \bar{L}_j^{o^2} - \bar{D}_{ij}^2}{2} - \bar{L}_i^o \bar{L}_j^o \cos \sigma_{ij} = 0
\end{aligned} \tag{80}$$

Three points are necessary and sufficient to form three linear equations for the corrections ΔX_o, ΔY_o and ΔZ_o. In case the Δ-values are large, iteration becomes necessary due to the neglected second order terms. It is possible to base a least squares adjustment for an over-determined solution on formula (80). The corresponding observation equations are:

$$
\begin{aligned}
&+ \left[(X_o^o - X_i)\left(A_{ij} - \frac{B_{ij}}{2\bar{L}_i^{o2}}\right) + (X_o^o - X_j)\left(A_{ij} - \frac{B_{ij}}{2\bar{L}_j^{o2}}\right) \right] \Delta X_o \\
&+ \left[(Y_o^o - Y_i)\left(A_{ij} - \frac{B_{ij}}{2\bar{L}_i^{o2}}\right) + (Y_o^o - Y_j)\left(A_{ij} - \frac{B_{ij}}{2\bar{L}_j^{o2}}\right) \right] \Delta Y_o \\
&+ \left[(Z_o^o - Z_i)\left(A_{ij} - \frac{B_{ij}}{2\bar{L}_i^{o2}}\right) + (Z_o^o - Z_j)\left(A_{ij} - \frac{B_{ij}}{2\bar{L}_j^{o2}}\right) \right] \Delta Z_o \\
&+ \frac{1}{2} \left[(\bar{L}_i^{o2} + \bar{L}_j^{o2} - \bar{D}_{ij}^2)\, A_{ij} - B_{ij} \right] + (a_i v_i + a_i' v_i' + a_j v_j + a_j' v_j') = 0
\end{aligned}
\tag{81}
$$

where $A_{ij} = \dfrac{\bar{\ell}_i \bar{\ell}_j}{\bar{L}_i^o \bar{L}_j^o}$

$$B_{ij} = \bar{\ell}_i^2 + \bar{\ell}_j^2 - \bar{d}_{ij}^2$$

$$a_i = (\bar{\ell}_i - \bar{x}_p) \frac{B_{ij}}{2\bar{\ell}_i^2} - (\bar{\ell}_j - \bar{x}_p) \qquad a_i' = (\bar{\ell}_i' - \bar{y}_p) \frac{B_{ij}}{2\bar{\ell}_i'^2} - (\bar{\ell}_j' - \bar{y}_p)$$

$$a_j = (\bar{\ell}_j - \bar{x}_p) \frac{B_{ij}}{2\bar{\ell}_j^2} - (\bar{\ell}_i - \bar{x}_p) \qquad a_j' = (\bar{\ell}_j' - \bar{y}_p) \frac{B_{ij}}{2\bar{\ell}_j'^2} - (\bar{\ell}_i' - \bar{y}_p)$$

and where $\bar{\ell}_i^2$, $\bar{d}_{ij}^2$, $\bar{L}_i^{o2}$ and $\bar{D}_{ij}^2$ have the same meaning as in formula (80).

The system of observation equations (81) may be written as;

$$
\begin{array}{l}
a_1v_1 + a_1'v_1' + a_2v_2 + a_2'v_2' \qquad + A_1\Delta X_o + B_1\Delta Y_o + C_1\Delta Z_o - L_1 = 0 \\
b_1v_1 + b_1'v_1' \quad \cdot \quad \cdot \; + b_3v_3 + b_3'v_3' \qquad + A_2\Delta X_o + B_2\Delta Y_o + C_2\Delta Z_o - L_2 = 0 \\
c_1v_1 + c_1'v_1' \quad \cdot \quad \cdot \quad \cdot \quad \cdot \; + c_4v_4 + c_4'v_4' \qquad + A_3\Delta X_o + B_3\Delta Y_o + C_3\Delta Z_o - L_3 = 0 \\
\hdashline
(m-1)_1v_1 + (m-1)_1'v_1' \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \; \ldots + (n-1)_nv_n + (n-1)_n'v_n' + A_{(n-1)}\Delta X_o + B_{(n-1)}\Delta Y_o + C_{(n-1)}\Delta Z_o - L_{(n-1)} = 0 \\
\cdot \quad \cdot \; + m_2v_2 + m_2'v_2' + m_3v_3 + m_3'v_3' \qquad + A_m\Delta X_o + B_m\Delta Y_o + C_m\Delta Z_o - L_m = 0 \\
\cdot \quad \cdot \; + (m+1)_2v_2 + (n+1)_2'v_2' + \quad \cdot \quad \cdot \; + (m+1)_4v_4 + (n+1)_4'v_4' \qquad + A_{(n+1)}\Delta X_o + B_{(n+1)}\Delta Y_o + C_{(n+1)}\Delta Z_o - L_{(n+1)} = 0 \\
\hdashline
\cdot \quad \cdot \; + r_2v_2 + r_2'v_2' \quad \cdot \quad \cdot \quad \cdot \quad \cdot \; \ldots + r_nv_n + r_n'v_n' + A_r\Delta X_o + B_r\Delta Y_o + C_r\Delta Z_o - L_r = 0
\end{array}
\tag{82}
$$

Number of equations $r = 2m-3$

The v's and v''s are the corrections on the measured plate coordinates ℓ and ℓ', respectively. The sum of the squares of the v's and v''s = $[vv] + [v'v']$ must become a minimum. ΔX_o, ΔY_o and ΔZ_o are the three unknowns of the solution. If the number r of the observation equations (82) equals u, the system represents the unique solution, because all v and v' values become zero. In order to have a least squares solution, it is necessary that $r > u$. Following Helmert's line of thought for solving a problem of "conditioned observations with unknowns", we may express the unknowns as functions of the v's and v''s, in u of the r observation equations. By substituting these functions into the remaining (r-u) observation equations, the unknowns are eliminated and we obtain finally a system of (r-u) condition equations between the v's and v''s. Consequently, for a least squares adjustment, it is further necessary that $n > (r-u)$. The mean error of unit weight is then $m = \sqrt{\frac{[vv] + [v'v']}{r-u}}$.

Further, we know that the number (r-u) of independent condition equations which exists between the observations or the residuals must equal the number of observations in excess. Each control point gives rise to two observations ℓ and ℓ' and so we have for n-points 2n observations. Because 3 points, or 6 observations, are necessary and sufficient for the unique solution, the number of observations in excess equals 2n-6. Thus we have the relation $(r-u) = (r-3) = (2n-6)$. Consequently, the pyramid method for n-points leads to $r = 2n-3$ observation equations. To set up these equations, we may combine any two of the control points with each of the other control points according to formula (82).

Temporarily leaving the outlined approach, we will first consider a direct solution of the least squares adjustment based upon the r observation equations given with formulas (82). According to Helmert, such a solution may be obtained by determining the minimum of the function

$$
\begin{aligned}
&[vv]+[v'v'] - 2k_1(a_1v_1+a_1'v_1'+a_2v_2+a_2'v_2' \quad . \quad . \quad +A_1\Delta X_o+B_1\Delta Y_o+C_1\Delta Z_o-L_1) \\
&- \qquad (83) \\
&\qquad - 2k_r(\quad . \quad . \quad +r_2v_2+r_2'v_2'+r_nv_n+r_n'v_n'+A_r\Delta X_o+B_r\Delta Y_o+C_r\Delta Z_o-L_r)
\end{aligned}
$$

By setting the differential quotients of the function (83) for $v_1 .. v_n$, $v_1' .. v_n'$, ΔX_o, ΔY_o and ΔZ_o each equal to zero, we obtain (2n+3) equations which, together with the r observation equations (formulas (82)), are sufficient to determine the three unknowns, the r correlates k, and the 2n-corrections v and v'.

The differentiation for v's and v''s, respectively, gives:

$$\left.\begin{aligned}
v_1 &= a_1k_1 + b_1k_2 + c_1k_3 + \dots (n-1)_1k_{(n-1)} \\
v_1' &= a_1'k_1 + b_1'k_2 + c_1'k_3 + \dots (n-1)_1'k_{(n-1)} \\
v_2 &= a_2k_1 + n_2k_n + (n+1)_2k_{(n+1)} + \dots r_2k_r \\
v_2' &= a_2'k_1 + n_2'k_n + (n+1)_2'k_{(n+1)} + \dots r_2'k_r \\
v_3 &= b_3k_2 + n_3k_n \\
v_3' &= b_3'k_2 + n_3'k_n \\
v_4 &= c_4k_3 + (n+1)_4k_{(n+1)} \\
v_4' &= c_4'k_3 + (n+1)_4'k_{(n+1)} \\
&- - - - - - - - - - - - \\
v_n &= (n-1)_nk_{(n-1)} + r_nk_r \\
v_n' &= (n-1)_n'k_{(n-1)} + r_n'k_r
\end{aligned}\right\} \text{number} = 2n \qquad (84)$$

The differentiation for ΔX_o, ΔY_o and ΔZ_o gives:

$$\text{n = number}\left\{\begin{aligned}
&A_1k_1 + A_2k_2 + A_3k_3 + \dots A_{(n-1)}k_{(n-1)} + A_nk_n + A_{(n+1)}k_{(n+1)} + \dots A_rk_r = 0 \\
&B_1k_1 + B_2k_2 + B_3k_3 + \dots B_{(n-1)}k_{(n-1)} + B_nk_n + B_{(n+1)}k_{(n+1)} + \dots B_rk_r = 0 \\
&C_1k_1 + C_2k_2 + C_3k_3 + \dots C_{(n-1)}k_{(n-1)} + C_nk_n + C_{(n+1)}k_{(n+1)} + \dots C_rk_r = 0
\end{aligned}\right. \qquad (85)$$

Substituting the formulas (84) into (82), we obtain with formulas (85) the following normal equations:

$$
\begin{array}{llllllll}
\boxed{aa}k_1+[ab]k_2+[ac]k_3+\cdots & [a(n-1)]k_{(n-1)}+[an]k_n+ & [a(n+1)]k_{(n+1)}+\cdots & [ar]k_r+ & A_1\Delta X_o+ & B_1\Delta Y_o+ & C_1\Delta Z_o-L_1 & =0\\
\qquad \boxed{bb}k_2+[bc]k_3+\cdots & [b(n-1)]k_{(n-1)}+[bn]k_n+ & 0 \quad + & 0 \quad + & A_2\Delta X_o+ & B_2\Delta Y_o+ & C_2\Delta Z_o-L_2 & =0\\
\qquad\qquad \boxed{cc}k_3+\cdots & [c(n-1)]k_{(n-1)}+ \quad 0 \quad + & [c(n+1)]k_{(n+1)}+ & 0 \quad + & A_3\Delta X_o+ & B_3\Delta Y_o+ & C_3\Delta Z_o-L_3 & =0\\
\hline
 & \underline{[(n-1)(n-1)]}k_{(n-1)}+ \quad 0 \quad + & 0 \quad + & [(n-1)r]k_r+ & A_{(n-1)}\Delta X_o+ & B_{(n-1)}\Delta Y_o+ & C_{(n-1)}\Delta Z_o-L_{(n-1)} & =0\\
 & \qquad +\boxed{nn}k_n+ & [n(n+1)]k_{(n+1)}+\cdots & [nr]k_r+ & A_n\Delta X_o+ & B_n\Delta Y_o+ & C_n\Delta Z_o-L_n & =0\\
 & & \underline{[(n+1)(n+1)]}k_{(n+1)}+\cdots & [(n+1)r]k_r+ & A_{(n+1)}\Delta X_o+ & B_{(n+1)}\Delta Y_o+ & C_{(n+1)}\Delta Z_o-L_{(n+1)} & =0\\
\hline
 & & & \boxed{rr}k_r+ & A_r\Delta X_o+ & B_r\Delta Y_o+ & C_r\Delta Z_o-L_r & =0\\
 & & & & \underline{0} & 0 & 0 & =0\\
 & & & & & \underline{0} & 0 & =0\\
 & & & & & & \underline{0} & =0
\end{array}
\qquad (86)
$$

$\text{number} = r + u = 2n$

After the correlates $k_1 \ldots k_r$ are eliminated, a system of three normal equations remains for the three unknowns. This system can be used to determine the weight of the unknowns. Thus the direct solution leads to a system of normal equations which are obtained from the observation equations (82) without elaborate computing. The number of normal equations to be solved can be reduced by basing the least squares solution on such condition equations as exist between the v's and v''s only. As explained previously, there are (r-u) such equations and, consequently, we have (2n-6), instead of 2n, normal equations. Since the majority of spatial resection problems are based on 4 to 6 ground control points, only 2, 4 or 6 normal equations have to be solved.

We may rearrange the equations (82) in such a way that the first three equations represent the unique solution with respect to optimum geometrical conditions.

$$
\text{number } r = 2n+3
\left\{
\begin{array}{l}
+a_1v_1+a_1'v_1'+a_2v_2+a_2'v_2 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdots\cdot + A_1\Delta X_o + B_1\Delta Y_o + C_1\Delta Z_o - L_1 = 0 \\
+b_1v_1+b_1'v_1' \quad \cdot \quad \cdot \quad +b_3v_3+b_3'v_3' \quad \cdot \quad \cdot \quad \cdot \quad \cdots\cdot + A_2\Delta X_o + B_2\Delta Y_o + C_2\Delta Z_o - L_2 = 0 \\
\cdot \quad \cdot \quad c_2v_2 + c_2'v_2'+c_3v_3+c_3'v_3' \quad \cdot \quad \cdot \quad \cdot \quad \cdots\cdot + A_3\Delta X_o + B_3\Delta Y_o + C_3\Delta Z_o - L_3 = 0 \\
\hline
+d_1v_1+d_1'v_1' \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad + d_4v_4 + d_4'v_4' \quad \cdot \quad \cdots\cdot + A_4\Delta X_o + B_4\Delta Y_o + C_4\Delta Z_o - L_4 = 0 \\
\hdashline
+n_1v_1+n_1'v_1' \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdots\cdot + n_nv_n + n_n'v_n' + A_n\Delta X_o + B_n\Delta Y_o + C_n\Delta Z_o - L_n = 0 \\
\cdot \quad \cdot \quad +(n+1)_2v_2+(n+1)_2'v_2' \quad \cdot \quad \cdot \quad +(n+1)_4v_4+(n+1)_4'v_4' \quad \cdot \quad \cdots\cdot + A_{(n+1)}\Delta X_o + B_{(n+1)}\Delta Y_o + C_{(n+1)}\Delta Z_o - L_{(n+1)} = 0 \\
\hdashline
\cdot \quad \cdot \quad r_2v_2 + r_2'v_2' \quad \cdot \quad \cdot \quad \cdot \quad \cdots\cdot + r_nv_n + r_n'v_n' + A_r\Delta X_o + B_r\Delta Y_o + C_r\Delta Z_o - L_r = 0
\end{array}
\right. \qquad (87)
$$

In order to solve the first three equations for ΔX_o, ΔY_o and ΔZ_o, we obtain from formula (87):

$$
\begin{aligned}
A_1\Delta X_o + B_1\Delta Y_o + C_1\Delta Z_o &= L_1 - (a_1v_1 + a_1'v_1' + a_2v_2 + a_2'v_2') = W_1 \\
A_2\Delta X_o + B_2\Delta Y_o + C_2\Delta Z_o &= L_2 - (b_1v_1 + b_1'v_1' + b_3v_3 + b_3'v_3') = W_2 \\
A_3\Delta X_o + B_3\Delta Y_o + C_3\Delta Z_o &= L_3 - (c_2v_2 + c_2'v_2' + c_3v_3 + c_3'v_3') = W_3
\end{aligned}
\qquad (88)
$$

and $\Delta X_o = \dfrac{D_{\Delta X}}{D} \qquad \Delta Y_o = \dfrac{D_{\Delta Y}}{D} \qquad \Delta Z_o = \dfrac{D_{\Delta Z}}{D}$

where (89)

$$D = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \qquad D_{\Delta X} = \begin{vmatrix} W_1 & B_1 & C_1 \\ W_2 & B_2 & C_2 \\ W_3 & B_3 & C_3 \end{vmatrix} \qquad D_{\Delta Y} = \begin{vmatrix} A_1 & W_1 & C_1 \\ A_2 & W_2 & C_2 \\ A_3 & W_3 & C_3 \end{vmatrix} \qquad D_{\Delta Z} = \begin{vmatrix} A_1 & B_1 & W_1 \\ A_2 & B_2 & W_2 \\ A_3 & B_3 & W_3 \end{vmatrix}$$

or with formulas (88) and (89)

$$\begin{aligned} D_{\Delta X} &= \alpha_x v_1 + \alpha'_x v'_1 + \beta_x v_2 + \beta'_x v'_2 + \gamma_x v_3 + \gamma'_x v'_3 - \lambda_x \\ D_{\Delta Y} &= \alpha_y v_1 + \alpha'_y v'_1 + \beta_y v_2 + \beta'_y v'_2 + \gamma_y v_3 + \gamma'_y v'_3 - \lambda_y \\ D_{\Delta Z} &= \alpha_z v_1 + \alpha'_z v'_1 + \beta_z v_2 + \beta'_z v'_2 + \gamma_z v_3 + \gamma'_z v'_3 - \lambda_z \end{aligned} \tag{90}$$

where

$$\begin{array}{lll} \alpha_x = a_1 a_x + b_1 b_x & \alpha_y = a_1 a_y + b_1 b_y & \alpha_s = a_1 a_z + b_1 b_z \\ \alpha'_x = a'_1 a_x + b'_1 b_x & \alpha'_y = a'_1 a_y + b'_1 b_y & \alpha'_z = a'_1 a_z + b'_1 b_z \\ \beta_x = a_2 a_x + c_2 c_x & \beta_y = a_2 a_y + c_2 c_y & \beta_z = a_2 a_z + c_2 c_z \\ \beta'_x = a'_2 a_x + c'_2 c_x & \beta'_y = a'_2 a_y + c'_2 c_y & \beta'_z = a'_2 a_z + c'_2 c_z \\ \gamma_x = b_3 b_x + c_3 c_x & \gamma_y = b_3 b_y + c_3 c_y & \gamma_z = b_3 b_z + c_3 c_z \\ \gamma'_x = b'_3 b_x + c'_3 c_x & \gamma'_y = b'_3 b_y + c'_3 c_y & \gamma'_z = b'_3 b_z + c'_3 c_z \\ \lambda_x = a_x L_1 + b_x L_2 + c_x L_3 & \lambda_y = a_y L_1 + b_y L_2 + c_y L_3 & \lambda_z = a_z L_1 + b_z L_2 + c_z L_3 \end{array}$$

with

$$\begin{array}{lll} a_x = B_3 C_2 - B_2 C_3 & a_y = A_2 C_3 - A_3 C_2 & a_z = A_3 B_2 - A_2 B_3 \\ b_x = B_1 C_3 - B_3 C_1 & b_y = A_3 C_1 - A_1 C_3 & b_z = A_1 B_3 - A_3 B_1 \\ c_x = B_2 C_1 - B_1 C_2 & c_y = A_1 C_2 - A_2 C_1 & c_z = A_2 B_1 - A_1 B_2 \end{array}$$

Inserting formulas (89) in the (r-u) remaining condition equations (87), we obtain:

$$\text{number}=(r-u)=2n-6\left\{\begin{array}{l}
+d_1v_1+d_1'v_1' \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad + \; d_4v_4+ \; d_4'v_4' \quad \cdot \; \cdots \quad + \; A_4'D_{\Delta X}+ \; B_4'D_{\Delta Y}+ \; C_4'D_{\Delta Z}-L_4 \; =0 \\
\hdashline
+n_1v_1+n_1'v_1' \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \; \cdots +n_nv_n+n_n'v_n'+ \; A_n'D_{\Delta X}+ \; B_n'D_{\Delta Y}+ \; C_n'D_{\Delta Z}-L_n \; =0 \\
\quad \cdot \quad +(n+1)_2v_2+(n+1)_2'v_2' \quad \cdot \quad \cdot \quad +(n+1)_4v_4+(n+1)_4'v_4' \quad \cdot \; \cdots \quad +A_{(n+1)}'D_{\Delta X}+B_{(n+1)}'D_{\Delta Y}+C_{(n+1)}'D_{\Delta Z}-L_{(n+1)}=0 \\
\hdashline
\quad \cdot \quad \cdot \quad +r_2v_2+ \; r_2'v_2' \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \; \cdots +r_nv_n+r_n'v_n'+ \; A_r'D_{\Delta X}+ \; B_r'D_{\Delta Y}+ \; C_r'D_{\Delta Z}-L_r \; =0
\end{array}\right. \tag{91}$$

where $A_i' = \dfrac{A_i}{D}$ $\qquad B_i' = \dfrac{B_i}{D}$ $\qquad$ and $C_i' = \dfrac{C_i}{D}$

With equations (90) this system may be written as:

$$
\begin{array}{l}
v_1(A'_4\alpha_x + B'_4\alpha_y + C'_4\alpha_z + d_1) + v'_1(A'_4\alpha'_x + B'_4\alpha'_y + C'_4\alpha'_z + d'_1) \\
v_2(A'_4\beta_x + B'_4\beta_y + C'_4\beta_z) + v'_2(A'_4\beta'_x + B'_4\beta'_y + C'_4\beta'_z) \\
v_3(A'_4\gamma_x + B'_4\gamma_y + C'_4\gamma_z) + v'_3(A'_4\gamma'_x + B'_4\gamma'_y + C'_4\gamma'_z) \\
v_4 d_4 + v'_4 d'_4 \qquad -(L_4 + A'_4\lambda_x + B'_4\lambda_y + C'_4\lambda_z) = 0 \\
\hdashline
v_1(A'_n\alpha_x + B'_n\alpha_y + C'_n\alpha_z + n_1) + v'_1(A'_n\alpha'_x + B'_n\alpha'_y + C'_n\alpha'_z + n'_1) \\
v_2(A'_n\beta_x + B'_n\beta_y + C'_n\beta_z) + v'_2(A'_n\beta'_x + B'_n\beta'_y + C'_n\beta'_z) \\
v_3(A'_n\gamma_x + B'_n\gamma_y + C'_n\gamma_z) + v'_3(A'_n\gamma'_x + B'_n\gamma'_y + C'_n\gamma'_z) \\
v_n n_n + v'_n n'_n \qquad -(L_n + A'_n\lambda_x + B'_n\lambda_y + C'_n\lambda_z) = 0 \\
v_1(A'_{(n+1)}\alpha_x + B'_{(n+1)}\alpha_y + C'_{(n+1)}\alpha_z) + v'_1(A'_{(n+1)}\alpha'_x + B'_{(n+1)}\alpha'_y + C'_{(n+1)}\alpha'_z) \\
v_2(A'_{(n+1)}\beta_x + B'_{(n+1)}\beta_y + C'_{(n+1)}\beta_z + (n+1)_2) + v'_2(A'_{(n+1)}\beta'_x + B'_{(n+1)}\beta'_y + C'_{(n+1)}\beta'_z + (n+1)'_2) \\
v_3(A'_{(n+1)}\gamma_x + B'_{(n+1)}\gamma_y + C'_{(n+1)}\gamma_z) + v'_3(A'_{(n+1)}\gamma'_x + B'_{(n+1)}\gamma'_y + C'_{(n+1)}\gamma'_z) \\
v_4(n+1)_4 + v'_4(n+1)'_4 \qquad -(L_{(n+1)} + A'_{(n+1)}\lambda_x + B'_{(n+1)}\lambda_y + C'_{(n+1)}\lambda_z) = 0 \\
\hdashline
v_1(A'_r\alpha_x + B'_r\alpha_y + C'_r\alpha_z) + v'_1(A'_r\alpha'_x + B'_r\alpha'_y + C'_r\alpha'_z) \\
v_2(A'_r\beta_x + B'_r\beta_y + C'_r\beta_z + r_2) + v'_2(A'_r\beta'_x + B'_r\beta'_y + C'_r\beta'_z + r'_2) \\
v_3(A'_r\gamma_x + B'_r\gamma_y + C'_r\gamma_z) + v'_3(A'_r\gamma'_x + B'_r\gamma'_y + C'_r\gamma'_z) \\
v_n r_n + v'_n r'_n \qquad -(L_r + A'_r\lambda_x + B'_r\lambda_y + C'_r\lambda_z) = 0
\end{array}
\qquad (92)
$$

In general nomenclature the condition equations (92) are:

$$
\text{number } (r-u) = 2n-6 \left\{
\begin{array}{l}
\delta_1 v_1 + \delta_1' v_1' + \delta_2 v_2 + \delta_2' v_2' + \delta_3 v_3 + \delta_3' v_3' + \delta_4 v_4 + \delta_4' v_4' \cdot \cdot \cdots \cdot \cdot + w_4 = 0 \\
\hdashline
\nu_1 v_1 + \nu_1' v_1' + \nu_2 v_2 + \nu_2' v_2' + \nu_3 v_3 + \nu_3' v_3' \cdot \cdot \cdot \cdot \cdots + \nu_n v_n + \nu_n' v_n' + w_n = 0 \\
(\nu+1)_1 v_1 + (\nu+1)_1' v_1' + (\nu+1)_2 v_2 + (\nu+1)_2' v_2' + (\nu+1)_3 v_3 + (\nu+1)_3' v_3' + (\nu+1)_4 v_4 + (\nu+1)_4' v_4' \cdot \cdot \cdots \cdot \cdot + w_{(n+1)} = 0 \\
\hdashline
\rho_1 v_1 + \rho_1' v_1' + \rho_2 v_2 + \rho_2' v_2' + \rho_3 v_3 + \rho_3' v_3' \cdot \cdot \cdot \cdot \cdots + \rho_n v_n + \rho_n' v_n' + w_r = 0
\end{array}
\right. \qquad (93)
$$

The corresponding normal equations are:

$$
\left.
\begin{array}{lllll}
\underline{[\delta\delta]}\, k_4 + \cdot \cdot \cdot \cdot \cdot & [\delta\nu]\, k_n + & [\delta(\nu+1)]\, k_{(n+1)} + \cdot \cdot \cdot \cdot & [\delta\ \rho]\, k_r + w_4 & = 0 \\
& \vdots & \vdots & \vdots & \vdots \\
& \underline{[\nu\nu]}\, k_n + & [\nu(\nu+1)]\, k_{(n+1)} + \cdot \cdot \cdot \cdot & [\nu\ \rho]\, k_r + w_n & = 0 \\
& & \underline{[(\nu+1)(\nu+1)]}\, k_{(n+1)} + \cdot \cdot \cdot \cdot & [(\nu+1)\rho]\, k_r + w_{(n+1)} & = 0 \\
& & & \vdots & \vdots \\
& & & \underline{[\rho\ \rho]}\, k_r + w_r & = 0
\end{array}
\right\} \text{number } (r-u) = 2n-6 \qquad (94)
$$

After computation of $k_4 \ldots k_r$, the v's and v''s are obtained from formula (93) and the unknowns ΔX_o, ΔY_o, and ΔZ_o from formula (89) in connection with (90).

The mean error of unit weight is $m = \sqrt{\frac{[vv] + [v'v']}{r-u}} = \sqrt{\frac{[vv] + [v'v']}{2n-6}}$ (95); with the computational check $[vv] + [v'v'] = -[wk]$ (96). In order to obtain the mean errors of the unknowns, denoted by m_{x_0}, m_{y_0} and m_{z_0} we add to our system of condition equations ~~(93)~~, a set of fictitious equations which are obtained by omitting the λ-terms from formulas (90). Thus we have the final system of equations:

$$\left.\begin{array}{l}
\left.\begin{array}{l}
\delta_1 v_1 + \delta_1' v_1' + \delta_2 v_2 + \delta_2' v_2' + \delta_3 v_3 + \delta_3' v_3' + \delta_4 v_4 + \delta_4' v_4' \cdot \cdot \cdots \cdot \cdot + w_4 = 0 \\
\hline
\rho_1 v_1 + \rho_1' v_1' + \rho_2 v_2 + \rho_2' v_2' + \rho_3 v_3 + \rho_3' v_3' \cdot \cdot \cdot \cdots + \rho_n v_n + \rho_n' v_n' + w_r = 0
\end{array}\right\} \text{number} = r-u = 2n-6 \\
\hline
\left.\begin{array}{l}
\alpha_x v_1 + \alpha_x' v_1' + \beta_x v_2 + \beta_x' v_2' + \gamma_x v_3 + \gamma_x' v_3' = 0 \\
\alpha_y v_1 + \alpha_y' v_1' + \beta_y v_2 + \beta_y' v_2' + \gamma_y v_3 + \gamma_y' v_3' = 0 \\
\alpha_z v_1 + \alpha_z' v_1' + \beta_z v_2 + \beta_z' v_2' + \gamma_z v_3 + \gamma_z' v_3' = 0
\end{array}\right\} \text{number} = u = 3
\end{array}\right. \qquad (97)$$

and the final normal equations are now:

$$\begin{array}{l|l|l|l|l}
 & D_{\Delta X} & D_{\Delta Y} & D_{\Delta Z} & \\
\boxed{[\delta\delta]}\, k_4 \cdots\cdots [\delta\rho]\, k_r + w_4 & \delta_1\alpha_x + \delta_1'\alpha_x' + \delta_2\beta_x + \delta_2'\beta_x' + \delta_3\gamma_x + \delta_3'\gamma_x' & \delta_1\alpha_y + \delta_1'\alpha_y' + \delta_2\beta_y + \delta_2'\beta_y' + \delta_3\gamma_y + \delta_3'\gamma_y' & \delta_1\alpha_z + \delta_1'\alpha_z' + \delta_2\beta_z + \delta_2'\beta_z' + \delta_3\gamma_z + \delta_3'\gamma_z' & = 0 \\
\quad\vdots \qquad\qquad \vdots & \qquad\vdots & \qquad\vdots & \qquad\vdots & (98) \\
\qquad\quad \boxed{[\rho\rho]}\, k_r + w_r & \rho_1\alpha_x + \rho_1'\alpha_x' + \rho_2\beta_x + \rho_2'\beta_x' + \rho_3\gamma_x + \rho_3'\gamma_x' & \rho_1\alpha_y + \rho_1'\alpha_y' + \rho_2\beta_y + \rho_2'\beta_y' + \rho_3\gamma_y + \rho_3'\gamma_y' & \rho_1\alpha_z + \rho_1'\alpha_z' + \rho_2\beta_z + \rho_2'\beta_z' + \rho_3\gamma_z + \rho_3'\gamma_z' & = 0 \\
 & \alpha_x\alpha_x + \alpha_x'\alpha_x' + \beta_x\beta_x + \beta_x'\beta_x' + \gamma_x\gamma_x + \gamma_x'\gamma_x' & \alpha_y\alpha_y + \alpha_y'\alpha_y' + \beta_y\beta_y + \beta_y'\beta_y' + \gamma_y\gamma_y + \gamma_y'\gamma_y' & \alpha_z\alpha_z + \alpha_z'\alpha_z' + \beta_z\beta_z + \beta_z'\beta_z' + \gamma_z\gamma_z + \gamma_z'\gamma_z' & = 0
\end{array}$$

By reducing the normal equations (98), we obtain k_4 to k_r from the w-column. After the elimination of these correlates, the columns headed by $D_{\Delta X}$, $D_{\Delta Y}$ and $D_{\Delta Z}$ each reduce to a term, which, when divided by D^2, is equal to the weighting factor of the corresponding unknown. If we denote these quotients by $[\alpha\alpha]$, $[\beta\beta]$, and $[\gamma\gamma]$, referring to the unknowns X_o, Y_o and Z_o, respectively, the mean errors are obtained by

$$m_{x_o} = m\sqrt{[\alpha\alpha]}$$

$$m_{y_o} = m\sqrt{[\beta\beta]}$$

$$m_{z_o} = m\sqrt{[\gamma\gamma]}$$

Despite the complexity of the theory of this least squares application, the computation can be systematically arranged for hand and electronic computing devices. The computational work is simplified by the fact that the auxiliary quantities $\alpha_x \ldots \ldots \gamma_z'$, λ_x, λ_y and λ_z, as they appear in the final condition equations (92), are formed from the coefficients of the first three observation equations only.

THE COMPUTATION OF APPROXIMATE VALUES OF THE UNKNOWNS

The process of a least squares adjustment of measurements generally requires setting up observation equations. If these equations are not linear, it is conventional to linearize them by the application of the Taylor series, neglecting second and higher order terms. In such cases, the neglected terms may require an iteration process, depending upon the quality of the primary approximate values of the unknowns and upon the degree of convergence of the solution. In order to minimize the number of iterations, a set of approximate values is computed from the minimum number of measurements necessary and sufficient to solve the unique case of the specific solution. The measurements for this step are selected in accordance with optimum geometrical conditions.

With the availability of electronic computing devices it is possible to use an alternate procedure, namely, to carry out, instead of preliminary computation of close approximate values, a large number of iterations in the final computations. Such a procedure may be economical in many cases. However, in our case it may not lead to the desired result. The reasons are found in the geometrical characteristics of our problem causing the solution to converge slowly, or even to delay convergence by occasional divergence. It is evident that three factors may contribute to the reluctance to converge. The first of these is concerned

with the geometrical condition of the specific case (presence of the "critical cylinder"). (See below). The second factor involves the linearization of the observation equations which has the effect of forcing the rotational movements from the actual curves into tangential displacements. The third factor originates from the fact that in the solution some of the unknowns are mutually dependent. Specifically, spatial translations perpendicular to the direction of the optical axis are correlated to the corresponding components of rotation, where the degree of correlation depends on the opening angle of the bundle of rays under consideration. These effects are shown by the computed mean errors of the unknowns in examples No. 2 and No. 3, on pages 72 and 83, respectively. In both cases, the mean errors of the translations in the direction of the optical axis, denoted by c and Z, are small, as is the mean error of the swing angle κ. However, the linear parameters perpendicular to the direction of the optical axis, denoted by $\overline{x}_p$, $\overline{y}_p$ and X_o, Y_o, respectively, as well as the corresponding rotational movements α and ω, show considerable larger mean errors.

From these considerations we may conclude that:

1) It is advisable to introduce sufficiently accurate approximate values in the final solution in order to limit the number of iterations.

2) It is preferable to compute these approximate values by first making a unique solution for the three translations and then computing the three rotations. In this manner, the solution is deprived of the compensating effect of the translations and rotations and will in each of these steps converge faster.

In aerial problems, the elements of interior orientation of the camera are known or may be determined from an independent camera calibration. The problem now is to compute sufficiently accurate approximate values for the spatial position of the nodal point and the corresponding rotational components. With the position known, the rotational components may be computed by the general formulas (67-71) by setting the Δ -values for all translation components equal to zero. The computation of the spatial position of the nodal point is the well known problem of the unique resection in space, as discussed at the beginning of the preceding chapter.

According to Finsterwalder[9] each set of three reference points determines for the spatial resection problem a critical surface, which is analytically expressed by a circular cylinder which contains the reference points and is normal to the plane of the three points under consideration. If the position of the center of projection is situated on or in the vicinity of the "critical cylinder", independent of the approximate values the solution will lead to unreliable or even impossible computational steps.

[9] See (4)

It is not planned to discuss, within the scope of this report, error theoretical problems in connection with the orientation of a photogrammetric camera. However, it should be mentioned that there is no substantial reason to consider the existence of the "critical cylinder" as a serious limitation in the practical application of a single vertical photograph for a spatial resection. It has become an almost stereotyped phrase in all textbooks that the spatial resection is worthless for practical purposes, due to the fact that in vertical, or approximately vertical, photographs the "crytical cylinder" always effects the result. The fact is that, for a vertical photograph, the coefficient determinant of the observation equations or of the corresponding normal equations asymptotically approaches zero for a decreasing opening angle σ of the surveying lens. For an unique solution based on an equilateral triangle, this determinant equals, for a vertical photograph, $D = 2c^6 \tan^9 \frac{\sigma}{2}$, indicating the influence of σ on the reliability of the resection solution. On the other hand, for $\sigma = 60^o$ or 90^o the determinant D will in such cases always be sufficiently different from zero. Because the forementioned determinant changes sign as the center of projection is moved from the inside of the "critical cylinder" to the outside, there will be a zone of unreliability of the solution in the vicinity of the "critical cylinder". However, it is readily seen that for vertical or approximately vertical photographs, taken with a 60^o or 90^o lens, the topography of the area being photographed will seldom cause the location of the center of projection to be located on or in the vicinity of the "critical cylinder".

In problems concerned with ground-based cameras, the location of the nodal point is generally known and the three elements of interior orientation of the cameras are usually given with sufficient approximation. Consequently, the problem is to compute sufficiently accurate approximate values for the rotational components. This may be done by making a unique solution using the general formulas (67-71), setting the Δ-values for all translation components equal to zero. In case the rotational components are known with a sufficient degree of approximation from previous experiments or from instrument dial readings, the general solution will lead directly, by a few iterations, to the desired result.

NUMERICAL EXAMPLES

Example 1. (Camera calibration from star images).

LEAST SQUARES ADJUSTMENT BASED ON COEFFICIENTS AS GIVEN IN FORMULA (2)

We use the following approximations computed from formulas (18):

$$a_1'^{o} = -.29012911 \qquad a_2'^{o} = -.08438538 \qquad a_o'^{o} = +.35178637$$

$$b_1'^{o} = -.07923629 \qquad b_2'^{o} = +.30874627 \qquad b_o'^{o} = +.09515064$$

$$c_1'^{o} = +.10950659 \qquad c_2'^{o} = -.00017082$$

and the auxiliaries from formulas (59) are:

$$a' = +.03357128$$

$$b' = -1.05440417$$

$$c' = +.34784335$$

$$d' = +.10176060$$

The approximation values satisfy the condition equations (27) 1) +.00000004 2) +.00000020

	Computed Reference Coordinates X meter	Y meter	① $a_1'^{o}X+b_1'^{o}Y+c_1'^{o}$	② $a_2'^{o}X+b_2'^{o}Y+c_2'^{o}$	③ $a_o'^{o}X+b_o'^{o}Y+1$
1	+.40758996	-.14685391	+.00288904	-.07990605	+1.12941135
2	+.53446718	-.05342108	-.04132501	-.06176560	+1.18293522
3	+.15447008	-.07298023	+.07047301	-.03573821	+1.04739635
4	+.31660935	+.06254320	+.01269331	-.00757804	+1.11732988
5	+.12239199	+.09272509	+.06664992	+.01812961	+1.05187869
6	+.43609833	+.07995540	-.02335360	-.01228521	+1.16102126
7	+.51791371	+.13719873	-.05162637	-.00151557	+1.19524953
8	+.32375583	+.18045918	+.00127668	+.02822502	+1.13106369
9	+.23371182	+.26993743	+.02031115	+.06344949	+1.10790135
10	+.50132793	+.32577947	-.06175679	+.05810763	+1.20735846

Measured Plate Coordinates			④	⑤	①-④	②-⑤	
	ℓ	ℓ'	ℓ ③	ℓ' ③	$-\Delta\ell$	$-\Delta\ell'$	p
1	+.002560	-.070772	+.00289129	-.07993070	-.00000225	+.00002465	.7840
2	-.034927	-.052237	-.04131638	-.06179299	-.00000863	+.00002739	.7146
3	+.067286	-.034119	+.07047511	-.03573612	-.00000210	-.00000209	.9115
4	+.011358	-.006780	+.01269063	-.00757550	+.00000268	-.00000254	.8010
5	+.063358	+.017234	+.06664493	+.01812808	+.00000499	+.00000153	.9038
6	-.020115	-.010581	-.02335394	-.01228477	+.00000034	-.00000044	.7419
7	-.043191	-.001275	-.05162402	-.00152394	-.00000235	+.00000837	.7000
8	+.001122	+.024948	+.00126905	+.02821778	+.00000763	+.00000724	.7817
9	+.018325	+.057266	+.02030229	+.06344508	+.00000886	+.00000441	.8147
10	-.051164	+.048123	-.06177329	+.05810171	+.00001650	+.00000592	.6860

The observation equations (60) are:

$(\Delta b_1')$	$(\Delta b_2')$	$(\Delta c_1')$	$(\Delta c_2')$	$(\Delta a_0')$	$(\Delta b_0')$	$-\Delta \ell$
-.13317059	-.42976455	+1.14177746	+.04147660	-.00104343	+.00037595	-.00000225
-.03547833	-.56354442	+1.18591085	+.05438770	+.01866734	-.00186584	-.00000863
+.06779447	-.16287390	+1.05373139	+.01571897	-.01039367	+.00491055	-.00000210
+.07317218	-.33383422	+1.11013046	+.03221836	-.00359605	-.00071037	+.00000268
+.09683395	-.12905062	+1.04257324	+.01245468	-.00775451	-.00587488	+.00000499
+.09459578	-.45982390	+1.15169390	+.04437763	+.00877212	+.00160830	+.00000034
+.15458576	-.54609038	+1.18015284	+.05270321	+.02236921	+.00592575	-.00000235
+.19132808	-.34136950	+1.11261631	+.03294559	-.00036325	-.00020248	+.00000763
+.27778343	-.24642672	+1.08129510	+.02378266	-.00428277	-.00494660	+.00000886
+.34260969	-.52860226	+1.17438359	+.05101543	+.02564994	+.01666818	+.00001650
+.42976455	-.13317059	-.04147660	+1.14177746	+.02884596	-.01039314	+.00002465
+.56354442	-.03547833	-.05438770	+1.18591085	+.02791896	-.00279056	+.00002739
+.16287390	-.06779447	-.01571897	+1.05373139	+.00527036	-.00249001	-.00000209
+.33383422	+.07317218	-.03221836	+1.11013046	+.00214661	+.00042404	-.00000254
+.12905062	+.09683395	-.01245468	+1.04257324	-.00210930	-.00159802	+.00000153
+.45982390	+.09459578	-.04437763	+1.15169390	+.00461436	+.00084601	-.00000044
+.54609038	+.15458576	-.05270321	+1.18015284	+.00066034	+.00017493	+.00000837
+.34136950	+.19132808	-.03294559	+1.11261631	-.00807706	-.00450210	+.00000724
+.24642672	+.27778343	-.02378266	+1.08129510	-.01338374	-.01545824	+.00000441
+.52860226	+.34260969	-.05101543	+1.17438359	-.02412540	-.01567749	+.00000592

The normal equations (62) are:

$\Delta b_1'$	$\Delta b_2'$	$\Delta c_1'$	$\Delta c_2'$	$\Delta a_o'$	$\Delta b_o'$	$-L$
+1.4252397	0	+.7219499	+3.2484484	+.0177384	-.0120319	+.000034509234
	+1.4252397	-3.2484484	+.7219499	-.0372606	-.0136234	-.000004575907
		+9.8451773	0	+.0359535	+.0138874	+.000019262371
			+9.8451773	+.0229839	-.0438053	+.000064499348
				+.0031415	+.0005459	+.000000983773
					+.0007713	-.000000281905
						+.000000001543

The general solution of the normal equations gives the following weighting coefficients:

[αα] = -5.2496817	+6.9326889	+2.3888838	+1.0653658	+75.9930602	+4.2702078
+6.9326889	[ββ] = -28.3565122	-8.8373739	+0.3345404	-273.0771170	-21.3270208
+2.3888838	-8.8373739	[γγ] = -2.8759711	+0.0352950	-84.7744432	-5.0434180
+1.0653658	+0.3345404	+0.0352950	[δδ] = -0.5330968	+3.3129443	-10.7288684
+75.9930602	-273.0771170	-84.7744432	+3.3129443	[εε] = -3085.6070599	+260.4505771
+4.2702078	-21.3270208	-5.0434180	-10.7288684	+260.4505771	[ζζ] = -2309.4688222

$$\sqrt{[\alpha\alpha]} = \pm 2.2912184 \qquad \sqrt{[\delta\delta]} = \pm 0.7301348$$

$$\sqrt{[\beta\beta]} = \pm 5.3250833 \qquad \sqrt{[\epsilon\epsilon]} = \pm 55.5482408$$

$$\sqrt{[\gamma\gamma]} = \pm 1.6958688 \qquad \sqrt{[\zeta\zeta]} = \pm 48.0569332$$

The solution gives the following unknowns:

$\Delta b_1' = -0.00002460 \qquad \Delta c_1' = -0.00001222 \qquad \Delta a_o' = -0.00065620$

$\Delta b_2' = -0.00004229 \qquad \Delta c_2' = +0.00000781 \qquad \Delta b_o' = +0.00036307$

From formula (58):

$$\Delta a_1' = +0.00004030 \qquad \Delta a_2' = -0.00002340$$

The final plate constants (51) are:

	Mean error of unknowns:	
$a_1' = -\,0.29008881$	$\pm\, 0.00002748$ [10]	
$b_1' = -\,0.07926089$	$\pm\, 0.00001237 = m\sqrt{[\alpha\alpha]}$	
$c_1' = +\,0.10949437$	$\pm\, 0.00000916 = m\sqrt{[\gamma\gamma]}$	
$a_2' = -\,0.08440878$	$\pm\, 0.00001224$ [10]	where
$b_2' = +\,0.30870398$	$\pm\, 0.00002876 = m\sqrt{[\beta\beta]}$	$m = \pm\, 5.4\mu$
$c_2' = -\,0.00016301$	$\pm\, 0.00000394 = m\sqrt{[\delta\delta]}$	
$a_0' = +\,0.35113018$	$\pm\, 0.00030000 = m\sqrt{[\epsilon\epsilon]}$	
$b_0' = +\,0.09551371$	$\pm\, 0.00025951 = m\sqrt{[\zeta\zeta]}$	

m = mean error of an observation of unit weight after the adjustment by means of formula (63).

The final plate constants satisfy the condition equations (27).

1) $+\,0.00000004$

2) $-\,0.00000019$

[10] The mean errors in a_1' and a_2' were computed as mean errors of a function of the unknowns, corresponding to formula (58).

Computation of the residuals by formulas (50-51)

	ρ microns	ρ' microns	v microns	v' microns	Final Check $\bar{x}-(\ell+v)$ 8th decimal place	$\bar{y}-(\ell'+v')$
1	+ 6.39	+ 6.44	+ 5.66	+ 5.70	0	-1
2	-10.92	+ 5.62	- 9.23	+ 4.75	-1	+1
3	+ 2.30	+ 0.83	+ 2.20	+ 0.79	0	0
4	+ 3.78	- 6.03	+ 3.38	- 5.40	0	-1
5	- 1.62	+ 3.36	- 1.54	+ 3.20	0	0
6	- 1.44	- 8.93	- 1.24	- 7.69	0	-2
7	- 9.60	- 2.10	- 8.04	- 1.76	0	0
8	+ 4.18	+ 3.51	+ 3.70	+ 3.10	0	0
9	+ 0.43	- 1.49	+ 0.39	- 1.35	0	+1
10	+ 5.69	- 1.63	+ 4.71	- 1.35	0	-1

$[vv] = 408.6$
$[pLL.6] = 408.7$

Final orientation elements computed from final plate constants by formulas (28) - (33):

$\kappa = 14^\circ\ 23'\ 58.8''$ $\qquad c = + 0.300743$ m

$\alpha = 19^\circ\ 20'\ 51.8''$ $\qquad \bar{x}_p = + 0.000057$ m

$\omega = +5^\circ\ 8'\ 58.4''$ $\qquad \bar{y}_p = + 0.000279$ m

Example 2. (Same camera calibration as Example 1)

LEAST SQUARES ADJUSTMENT OF THE GENERAL SOLUTION BASED ON FORMULA (48)

Given:

$X_o = 0$

$Y_o = 0$

$Z_o = 0$ and $Z = 1$

Approximation values of the unknowns:

$\kappa^o = 14^o \ 23'$ $\qquad c^o = 0.300500$ m

$\alpha^o = 19^o \ 23'$ $\qquad \bar{x}_p^o = 0.000000$

$\omega^o = 5^o \ 8'$ $\qquad \bar{y}_p^o = 0.000000$

$A^o = -0.90637482$ $\qquad A'^o = -0.26309254$ $\qquad D^o = +0.33055561$

$B^o = -0.24741181$ $\qquad B'^o = +0.96477036$ $\qquad E^o = +0.08947375$

$C^o = +0.34245013$ $\qquad C'^o = +0.00068661$ $\qquad F^o = +0.93953576$

	Computed Reference Coordinates X meter	Y meter	m from (67)	n	q
1	+.40758996	-.14685391	+.00935425	-.24822757	+1.06112734
2	+.53446718	-.05342108	-.12876046	-.19146679	+1.11142710
3	+.15447008	-.07298023	+.22049851	-.11036248	+ .98406690
4	+.31660935	+.06254320	+.04000946	-.02227112	+1.04978873
5	+.12239199	+.09272500	+.20857583	+.05794461	+ .98828958
6	+.43609833	+.07995540	-.07260033	-.03690901	+1.09084442
7	+.51791372	+.13719873	-.16091840	-.00320736	+1.12301073
8	+.32375583	+.18045918	+.00435827	+.08961053	+1.06270143
9	+.23371182	+.26993743	+.06383391	+.19962641	+1.04094283
10	+.50132793	+.32577947	-.19254257	+.18109335	+1.13440123

	from (67)		Measured Plate Coordinates		from (67)	
	$\ell^\circ = +c'$	$\ell'^\circ = -c$	ℓ	ℓ'	$-\Delta\ell^\circ = \ell^\circ - \ell$	$-\Delta\ell' = \ell'^\circ - \ell'$
1	+.00264902	-.07029541	+.002560	-.070772	+.00008902	+.00047659
2	-.03481337	-.05176747	-.034927	-.052237	+.00011363	+.00046953
3	+.06733262	-.03370088	+.067286	-.034119	+.00004662	+.00041812
4	+.01145263	-.00637507	+.011358	-.006780	+.00009463	+.00040493
5	+.06341971	+.01761868	+.063358	+.017234	+.00006171	+.00038468
6	-.01999955	-.01016750	-.020115	-.010581	+.00011545	+.00041350
7	=.04305923	-.00085824	-.043191	-.001275	+.00013177	+.00041676
8	+.00123239	+.02533916	+.001122	+.024948	+.00011039	+.00039116
9	+.01842761	+.05762827	+.018325	+.057266	+.00010261	+.00036227
10	-.05100404	+.04850096	-.051164	+.048123	+.00015996	+.00037796

	x°	y°	a	b	d
	from (67)		from (67) and (68)		
1	+.01489596	-.06875007	+.29607230	+.07525270	+.00881537
2	+.04658162	-.04149692	+.29992023	+.06983916	-.11585148
3	-.05685054	-.04937051	+.30561617	+.08570902	+.22406862
4	-.00951003	-.00902017	+.29084489	+.07499042	+.03811191
5	-.06580847	+.00131246	+.30217008	+.07436965	+.21104729
6	+.02189836	-.00488075	+.29227480	+.07432181	-.06655424
7	+.04192275	+.00986492	+.29597338	+.07606020	-.14329195
8	-.00748821	+.02423878	+.28767689	+.07454723	+.00410113
9	-.03216534	+.05124437	+.28672184	+.07150417	+.06132316
10	+.03735731	+.05965054	+.29188918	+.08477116	-.16973058

	a'	b'	d'
		from (67) and (68)	
1	+.07805487	-.30716354	-.23392815
2	+.07922485	-.29822969	-.17227111
3	+.07402157	-.29661784	-.11214935
4	+.07517102	-.29127233	-.02121188
5	+.08386460	-.29115792	+.05863121
6	+.07329578	-.29124611	-.03383527
7	+.07061383	-.29105280	-.00285604
8	+.07508641	-.29312486	+.08432333
9	+.08213980	-.30090834	+.19177461
10	+.06377841	-.30070862	+.16140087

$$\frac{[a]}{n} = +\,0.29491598 \qquad \frac{[a']}{n} = +\,0.07552511$$

$$\frac{[b]}{n} = +\,0.07613655 \qquad \frac{[b']}{n} = -\,0.29614820$$

$$\frac{[c]}{n} = +\,0.00240775 \qquad \frac{[c']}{n} = +\,0.00156378$$

$$\frac{[d]}{n} = +\,0.00520392 \qquad \frac{[d']}{n} = -\,0.00801248$$

$$\frac{[\Delta\ell]}{n} = -\,0.00010258 \qquad \frac{[\Delta\ell']}{n} = -\,0.00041155$$

The reduced observation equations from (70) are:[11]

	(a)	(b)	(c)	(d)	$-L$
	$\Delta\alpha'$	$\Delta\omega'$	$\Delta\kappa'$	Δc	
	$\Delta\alpha' = \frac{\Delta\alpha}{100}$	$\Delta\omega' = \frac{\Delta\omega}{100}$	$\Delta\kappa' = \frac{\Delta\kappa}{10}$		
1	+.115632	-.088385	+.6788766	+.00361145	-.00001356
2	+.500425	-.629739	+.4935972	-.12105540	+.00001105
3	+1.070019	+.957247	+.3129313	+.21886470	-.00005596
4	-.407109	-.114613	+.0396732	+.03290799	-.00000795
5	+.725410	-.176690	-.2002643	+.20584337	-.00004087
6	-.264118	-.181474	+.0775975	-.07175816	+.00001287
7	+.105740	-.007635	-.0154951	-.14849587	+.00002919
8	-.723909	-.158932	-.2774691	-.00110279	+.00000781
9	-.819414	-.463238	-.6003602	+.05611924	+.00000003
10	-.302680	+.863461	-.5090871	-.17493450	+.00005738
1	+.252976	-1.101534	+.0108524	-.22591567	+.00006504
2	+.369974	-.208149	-.3637715	-.16425863	+.00005798
3	-.150354	-.046964	+.6576884	-.10413687	+.00000657
4	-.035409	+.487587	+.0988885	-.01320240	-.00000662
5	+.833949	+.499028	+.6185593	+.06664369	-.00002687
6	-.222933	+.490209	-.2156333	-.02582279	+.00000195
7	-.491128	+.509540	-.4462301	+.00515644	+.00000521
8	-.043870	+.302334	-.0033139	+.09233581	-.00002039
9	+.661469	-.476014	+.1686383	+.19978709	-.00004928
10	-1.174670	-.456042	-.5256782	+.16941335	-.00003359

[11] The decimal point in the coefficients (a)...(c) was moved for the convenience of the numerical computations.

The combined normal equations by means of (71) are:

$\Delta\alpha'$	$\Delta\omega'$	$\Delta\kappa'$	Δc	$-L$
$\underline{+6.4991619}$	+.7961320	+2.6017281	+.2070662	-.000086182255
	$\underline{+5.1550494}$	+.0433678	+.2428437	-.000069898434
		$\underline{+3.0547725}$	0	-.000070914325
			$\underline{+.3382906}$	-.000081658617
				$\underline{+.000000021947}$

The weighting coefficients from the general solution of the normal equations are:

$[\alpha'\alpha'] = \underline{-.2449965}$	+.0300362	+.2082353	+.1283995	$\sqrt{[\alpha\alpha]} = \pm\ 49.5$
+.0300361	$[\beta'\beta'] = \underline{-.2044813}$	-.0226786	+.1284029	$\sqrt{[\beta\beta]} = \pm\ 45.2$
+.2082353	-.0226786	$[\gamma'\gamma'] = \underline{-.5043872}$	-.1111798	$\sqrt{[\gamma\gamma]} = \pm\ 7.1$
+.1283995	+.1284029	-.1111798	$[\delta\delta] = \underline{-3.1268057}$	$\sqrt{[\delta\delta]} = \pm\ 1.8$

The solution gives the following unknowns:

$$\Delta\alpha = -\ 0.00062369 = -\ 2'\ 08.65''$$

$$\Delta\omega = +\ 0.00028274 = +58.32''$$

$$\Delta\kappa = +\ 0.00028486 = +58.76''$$

$$\Delta c = +\ 0.00024317$$

From formula (69):

$$\Delta\bar{x}_p = +\ 0.00005788$$

$$\Delta\bar{y}_p = -\ 0.00027921$$

The final orientation elements from (66) are:

	Mean errors of unknowns
$\kappa = 14^\circ\ 23'\ 58.8''$	$\pm 7.9'' = \rho m \sqrt{[\gamma\gamma]}$
$\alpha = 19^\circ\ 20'\ 51.4''$	$\pm 55.1'' = \rho m \sqrt{[\alpha\alpha]}$
$\omega = 5^\circ\ 08'\ 58.3''$	$\pm 50.4'' = \rho m \sqrt{[\beta\beta]}$
$c = 0.300743$	$\pm .0000097 = m \sqrt{[\delta\delta]}$
$\bar{x}_p = +0.000058$	$\pm .000078$ [12]
$\bar{y}_p = -0.000279$	$\pm .000078$ [12]

where

$m = \pm 5.4\mu$

and

$\rho = 206265''$

m = mean error of an observation of unit weight after the adjustment by formula (72).

Computations of the residuals by formula (68):

	v microns	v' microns	Final Check by (64) $G-(\ell+v)$ 8th decimal place	$G'-(\ell'+v')$
1	+ 5.69	+ 5.72	-10	-17
2	- 9.23	+ 4.78	-11	-17
3	+ 2.21	+ 0.79	+ 8	-15
4	+ 3.40	- 5.41	-10	-15
5	- 1.54	+ 3.17	- 8	-14
6	- 1.24	- 7.70	-10	-15
7	- 8.04	- 1.74	-11	-15
8	+ 3.70	+ 3.10	-10	-14
9	+ 0.38	- 1.36	- 9	-14
10	+ 4.67	- 1.33	-11	-13

$[vv] = 409.2 \qquad [pLL.6] = 409.6$

12 The mean errors in $\bar{x}_p$ and $\bar{y}_p$ were computed as mean errors of a function of the unknowns corresponding to formulas (69).

A comparison of the results obtained from examples No. 1 and No. 2 shows a sufficient agreement between the two methods.

FINAL RESULT

	Method "A" based on formulas (2)	Method "B" based on formulas (48)
α	$+19^\circ\ 20'\ 51.8''$	$+19^\circ\ 20'\ 51.4''$
ω	$+\ 5^\circ\ 08'\ 58.4''$	$+\ 5^\circ\ 08'\ 58.3''$
κ	$+14^\circ\ 23'\ 58.8''$	$+14^\circ\ 23'\ 58.8''$
c	+ 0.300743	+ 0.300743
$\bar{x}_p$	+ 0.000057	+ 0.000058
$\bar{y}_p$	− 0.000279	− 0.000279

RESIDUALS

Method "A" v microns	Method "A" v' microns	Method "B" v microns	Method "B" v' microns
+ 5.7	+ 5.7	+ 5.7	+ 5.7
− 9.2	+ 4.8	− 9.2	+ 4.8
+ 2.2	+ 0.8	+ 2.2	+ 0.8
+ 3.4	− 5.4	+ 3.4	− 5.4
− 1.5	+ 3.2	− 1.5	+ 3.2
− 1.2	− 7.7	− 1.2	− 7.7
− 8.0	− 1.8	− 8.0	− 1.7
+ 3.7	+ 3.1	+ 3.7	+ 3.1
+ 0.4	− 1.4	+ 0.4	− 1.4
+ 4.7	− 1.4	+ 4.7	− 1.3
	[vv] = 408.6		[vv] = 409.2

Example 3. (A spatial resection from an approximately vertical photograph.)

LEAST SQUARES ADJUSTMENT OF THE GENERAL SOLUTION BASED ON FORMULA (48)

The computation follows purposely such numerical steps as are suitable for electronic computing devices, given in formulas (73) to (75).

Elements of interior orientation as obtained from a camera calibration, and assumed as free of errors:

$$c = 0.16999$$

$$\bar{x}_p = 0.00000$$

$$\bar{y}_p = 0.00000$$

First approximations of the unknowns taken directly from the photograph:

$$X_o^o = 550\text{m} \qquad \kappa^o = 66^o$$

$$Y_o^o = 640\text{m} \qquad \alpha^o = 180^o$$

$$Z_o^o = 2600\text{m} \qquad \omega^o = 0^o$$

Auxiliary unknowns from (73):

$$r^o = \sin\alpha^o = 0 \qquad r'^o = \cos\alpha^o = -1$$

$$s^o = \sin\omega^o = 0 \qquad s'^o = \cos\omega^o = +1$$

$$t^o = \sin\kappa^o = +.91354546 \qquad t'^o = \cos\kappa^o = +.40673664$$

Auxiliaries from formulas (73):

$$A^o = +.40673664 \qquad A'^o = +.91354546 \qquad D^o = 0$$

$$B^o = -.91354546 \qquad B'^o = +.40673664 \qquad E^o = 0$$

$$C^o = 0 \qquad C'^o = 0 \qquad F^o = -1$$

	Coordinates of ground control points			from (67)				
	X	Y	Z	m	n	q	$\ell^o = c'$	$\ell'^o = -c$
37	1218.45	290.8	446.1	+590.893182	+468.627028	+2153.90	+.04663445	-.03698496
38	750.95	125.8	435.9	+551.478803	-25.567020	+2164.10	+.04331865	+.00200829
39	273.20	0	440.1	+472.084392	-513.180833	+2159.90	+.03715432	+.04038873
46	1094.25	738.4	448.4	+131.473543	+537.220002	+2151.60	+.01038724	-.04244378
47	621.15	622.6	437.6	+44.835003	+57.921542	+2162.40	+.00352456	-.00455331
48	120.55	392.1	442.4	+51.794869	-493.152111	+2157.60	+.00408074	+.03885379
51	929.95	1177.3	470.0	-336.308389	+565.641194	+2130.00	-.02683994	-.04514242
52	446.05	1060.4	449.7	-426.334785	+76.029033	+2150.30	-.03370351	-.00601041
53	0	934.8	439.0	-493.018354	-382.544042	+2161.00	-.03878213	+.03009193

Measured plate coordinates		$-\Delta\ell =$	$-\Delta\ell' =$		
ℓ	ℓ'	$\ell^o - \ell$	$\ell'^c - \ell'$	x^o	y^o
+.45978	+.036240	+.00065645	+.00074496	-.05275538	-.02755955
+.043105	-.002307	+.00021365	+.00029871	-.01578462	-.04039040
+.037465	-.040100	-.00031068	-.00028873	+.02178492	-.05036974
+.010086	+.041152	+.00030124	+.00129178	-.04299919	+.00777423
+.003815	+.003612	-.00029044	+.00094131	-.00559322	-.00136785
+.004931	-.039019	-.00085026	+.00016521	+.03383492	-.01953120
-.026709	+.043294	-.00013094	+.00184842	-.03032287	+.04288058
-.032878	+.004552	-.00082551	+.00145841	+.00821767	+.03323434
-.037327	-.030990	-.00145513	+.00089807	+.04326446	+.02318975

$[\Delta l]$ +.00269162 $[\Delta l']$ -.00735814

discrimination factor $\dfrac{|[\Delta l]| + |[\Delta l']|}{2n}$ = .00055832

a	b	e=f'	f	g
+.08361388	+.16285418	-.00003210	+.00007210	-.00002165
+.07316356	+.16558630	-.00003195	+.00007176	-.00002002
+.06437968	+.16630279	-.00003201	+.00007190	-.00001720
+.07176863	+.15481855	-.00003213	+.00007218	-.00000483
+.06925713	+.15532195	-.00003197	+.00007182	-.00000163
+.06832893	+.15576245	-.00003205	+.00007198	-.00000189
+.06435344	+.16206406	-.00003246	+.00007291	+.00001260
+.07077046	+.16188288	-.00003215	+.00007222	+.00001567
+.07901167	+.16058418	-.00003199	+.00007186	+.00001795

a'	b'	e'	g'
+.16677165	-.06314499	-.00007210	-.000[illegible]1717
+.15510711	-.06961834	-.00007176	+.000[illegible]0093
+.16046957	-.08110875	-.00007190	+.000[illegible]1870
+.16602980	-.07108226	-.00007218	-.000[illegible]1973
+.15544341	-.06910452	-.00007182	-.000[illegible]00211
+.16302707	-.07360531	-.00007198	+.000 01801
+.16334611	-.08052850	-.00007291	-.000[illegible]02119
+.15500303	-.07031624	-.00007222	-.000[illegible]00280
+.16295234	-.06503607	-.00007186	+.00[illegible]01393

The observation equations (74) are:[13]

$\Delta\alpha$	$\Delta\omega$	$\Delta\kappa' = \frac{\Delta\kappa}{10}$	$\Delta X_o' = \frac{\Delta X_o}{10000}$	$\Delta Y_o' = \frac{\Delta Y_o}{10000}$	$\Delta Z_o' = \frac{\Delta Z_o}{10000}$	$-L$
+.08361388	+.16285418	-.3698496	-.3210	+.7210	-.2165	+.00065645
+.07316356	+.16558630	+.0200829	-.3195	+.7176	-.2002	+.00021365
+.06437968	+.16630279	+.4038873	-.3201	+.7190	-.1720	-.00031068
+.07176863	+.15481855	-.4244378	-.3213	+.7218	-.0483	+.00030124
+.06925713	+.15532195	-.0455331	-.3197	+.7182	-.0163	-.00029044
+.06832893	+.15576245	+.3885379	-.3205	+.7198	-.0189	-.00085026
+.06435344	+.16206406	-.4514242	-.3246	+.7291	+.1260	-.00013094
+.07077046	+.16188288	-.0601041	-.3215	+.7222	+.1567	-.00082551
+.07901167	+.16058418	+.3009193	-.3199	+.7186	+.1795	-.00145513
+.16677165	-.06314499	+.4663445	-.7210	-.3210	-.1717	+.00074496
+.15510711	-.06961834	+.4331865	-.7176	-.3195	+.0093	+.00029871
+.16046957	-.08110875	+.3715432	-.7190	-.3201	+.1870	-.00028873
+.16602980	-.07108226	+.1038724	-.7218	-.3213	-.1973	+.00129178
+.15544341	-.06910452	+.0352456	-.7182	-.3197	-.0211	+.00094131
+.16302707	-.07360531	+.0408074	-.7198	-.3205	+.1801	+.00016521
+.16334611	-.08052850	-.2683994	-.7291	-.3246	-.2119	+.00184842
+.15500303	-.07031624	-.3370351	-.7222	-.3215	-.0280	+.00145841
+.16295234	-.06503607	-.3878213	-.7186	-.3199	+.1393	+.00089807

[13]The decimal point in the coefficients $\Delta\kappa'$, $\Delta X_o'$, $\Delta Y_o'$ and $\Delta Z_o'$ was moved for the convenience of the numerical computations.

The normal equations are:

$\Delta\alpha$	$\Delta\omega$	$\Delta\kappa'$	$\Delta X_o'$	$\Delta Y_o'$	$\Delta Z_o'$	$-L$
+.2796869	-.0000153	+.0546345	-1.2507295	-.0001203	-.0364435	+.000998403801
	+.2785281	-.0685966	+.0001944	+1.2482496	-.0281270	-.000952080770
		+1.8243535	-.2470901	-.3227549	+.0000345	-.001930430958
			+5.6030266	0	+.1518194	-.004455105911
				+5.6030266	-.1116534	-.004304499046
					+.3935782	-.001263768873
						+.000013946912

Solution of the normal equations gives: and from (75)

$\Delta\alpha = +.01070070$ $\quad \Delta X_o = +31.566$ $\quad \Delta r = -.01070070$

$\Delta\omega = -.00356411$ $\quad \Delta Y_o = +17.033$ $\quad \Delta s = -.00356411$

$\Delta\kappa = +.01332341$ $\quad \Delta Z_o = +32.129$ $\quad \Delta t = +.00541912$

2nd approximations from formulas (66) and (73):

$X_o^o = 581.56$ $\quad r^o = -.01070070$ $\quad r'^o = \sqrt{1 - r^{o^2}} = -.99994275$

$Y_o^o = 657.03$ $\quad s^o = -.00356411$ $\quad s'^o = \sqrt{1 - s^{o^2}} = +.99999365$

$Z_o^o = 2632.13$ $\quad t^o = +.91896458$ $\quad t'^o = \sqrt{1 - t^{o^2}} = +.39434008$

$A^o = +.39435255$ $\quad A'^o = +.91889693$ $\quad D^o = -.01070063$

$B^o = -.91895874$ $\quad B'^o = +.39433758$ $\quad E^o = -.00356411$

$C^o = -.00094461$ $\quad C'^o = -.01123896$ $\quad F^o = -.99993640$

m	n	q	$\ell^0 = c'$	$\ell'^0 = -c$	$-\Delta\ell$	$-\Delta\ell'$
+589.774401	+465.386718	+2160.38	+.04598086	+.03628317	.00000286	+.00004317
+557.052411	-29.148661	+2196.17	+.04311749	-.00225619	+.00001249	+.00005081
+484.251522	-517.806540	+2197.53	+.03745929	-.04005494	-.00000571	+.00004506
+129.467709	+527.739370	+2177.81	+.01010566	+.04119295	+.00001966	+.00004095
+49.325142	+47.466321	+2194.09	+.00382153	+.00367752	+.00000653	+.00006552
+63.727711	-503.482241	+2195.47	+.00493428	-.03898343	+.00000328	+.00003557
-338.675809	+549.596607	+2156.41	-.02669785	+.04332475	+.00001115	+.00003075
-422.057556	+59.072470	+2182.30	-.03287612	+.00460144	+.00000188	+.00004944
-482.527186	-400.210049	+2198.22	-.03731419	-.03094854	+.00001281	+.00004146

$[\Delta\ell] = -.00006495$ $[\Delta\ell'] = -.00040273$

discrimination factor = .00002598

x^o	y^o	a	b	e	f	g
-.05147504	-.02794687	+.08108624	+.16377418	-.00003097	+.00007157	-.00002101
-.01492960	-.04051315	+.07081223	+.16649084	-.00003073	+.00007106	-.00001956
+.02203737	-.05021903	+.06203453	+.16728114	-.00003069	+.00007103	-.00001697
-.04183993	+.00695729	+.06966757	+.15580119	-.00003083	+.00007171	-.00000457
-.00488649	-.00206166	+.06715641	+.15626114	-.00003057	+.00007119	-.00000167
+.03387861	-.01990716	+.06591113	+.15679263	-.00003056	+.00007115	-.00000217
-.02928588	+.04161906	+.06258839	+.16275129	-.00003095	+.00007249	+.00001245
+.00873581	+.03202652	+.06873934	+.16240873	-.00003056	+.00007164	+.00001514
+.04315509	+.02208617	+.07639598	+.16106288	-.00003031	+.00007112	+.00001705

a'	b'	e'	f'	g'
+.16703683	-.06106881	-.00007182	-.00003080	-.00001576
+.15586198	-.06757158	-.00007111	-.00003052	+.00000190
+.16127295	-.07886704	-.00007089	-.00003044	+.00001910
+.16631661	-.06871980	-.00007193	-.00003085	-.00001804
+.15630589	-.06698927	-.00007121	-.00003056	-.00000081
+.16396546	-.07159913	-.00007096	-.00003047	+.00001863
+.16377289	-.07764117	-.00007265	-.00003116	-.00001920
+.15609452	-.06790079	-.00007160	-.00003072	-.00000123
+.16420359	-.06301284	-.00007091	-.00003044	+.00001495

The observation equations (74) are: [14]

$\Delta \alpha$	$\Delta \omega$	$\Delta \kappa' = \frac{\Delta \kappa}{10}$	$\Delta X'_o = \frac{\Delta X_o}{10000}$	$\Delta Y'_o = \frac{\Delta Y_o}{10000}$	$\Delta Z'_o = \frac{\Delta Z_o}{10000}$	-L
+.08108624	+.16377418	-.3628317	-.3097	+.7157	-.2101	+.00000286
+.07081223	+.16649084	+.0225619	-.3073	+.7106	-.1956	+.00001249
+.06203453	+.16728114	+.4005494	-.3069	+.7103	-.1697	-.00000571
+.06966757	+.15580119	-.4119295	-.3083	+.7171	-.0457	+.00001966
+.06715641	+.15626114	-.0367752	-.3057	+.7119	-.0167	+.00000653
+.06591113	+.15679263	+.3898343	-.3056	+.7115	-.0217	+.00000328
+.06258839	+.16275129	-.4332475	-.3095	+.7249	+.1245	+.00001115
+.06873934	+.16240873	-.0460144	-.3056	+.7164	+.1514	+.00000188
+.07639598	+.16106288	+.3094854	-.3031	+.7112	+.1705	+.00001281
+.16703683	-.06106881	+.4598086	-.7182	-.3080	-.1576	+.00004317
+.15586198	-.06757158	+.4311749	-.7111	-.3052	+.0190	+.00005081
+.16127295	-.07886704	+.3745929	-.7089	-.3044	+.1910	+.00004506
+.16631661	-.06871980	+.1010566	-.7193	-.3085	-.1804	+.00004095
+.15630589	-.06698927	+.0382153	-.7121	-.3056	-.0081	+.00006552
+.16396546	-.07159913	+.0493428	-.7096	-.3047	+.1863	+.00003557
+.16377289	-.07764117	-.2669785	-.7265	-.3116	-.1920	+.00003075
+.15609452	-.06790079	-.3287612	-.7160	-.3072	-.0123	+.00004944
+.16420359	-.06301284	-.3731419	-.7091	-.3044	+.1495	+.00004146

[14] The decimal point in the coefficients $\Delta\kappa'$, $\Delta X'_o$, $\Delta Y'_o$ and $\Delta Z'_o$ was moved for the convenience of the numerical computations.

The normal equations are:

$\Delta\alpha$	$\Delta\omega$	$\Delta\kappa'$	$\Delta X_o'$	$\Delta Y_o'$	$\Delta Z_o'$	$-L$
+.2789547	+.0000311	+.0638944	-1.2311895	-.0000846	-.0180243	+.000069434817
	+.2780665	-.0588336	-.0002980	+1.2289032	-.0369822	-.000017356762
		+1.7715662	-.2874000	-.2776242	+.0041309	+.000016044576
			+5.4427847	-.0010660	+.0764968	+.000307508970
				+5.4396875	-.1459643	-.000076917059
					+.3698930	+.000001866295
						+.0000000019748

The weight coefficients from the general solution of the normal equations are:

$[\alpha\alpha]$ = -2233.7381085	+183.3556248	-1.8612633	-505.3465789	-41.7170604	-2.4459707
+183.3556248	$[\beta\beta]$ = -2598.9118114	+5.9227910	+42.1574868	+586.6886728	-28.1769704
-1.8612633	+5.9227910	$[\gamma\gamma]$ = -.5884181	-0.4529206	-1.3665158	+0.0624646
-505.3465789	+42.1574868	-.4529206	$[\delta\delta]$ = -114.5124180	-9.5910103	-0.5073714
-41.7170604	+586.6886728	-1.3665158	-9.5910103	$[\epsilon\epsilon]$ = -132.6287780	+6.2866610
-2.4459707	-28.1769704	+0.0624646	-0.5073714	+6.2866610	$[\zeta\zeta]$ = -3.0548070

$\sqrt{[\alpha\alpha]} = \pm\ 47.262439$ $\qquad \sqrt{[\delta\delta]} = \pm\ 107010.475$

$\sqrt{[\beta\beta]} = \pm\ 50.979523$ $\qquad \sqrt{[\epsilon\epsilon]} = \pm\ 115164.568$

$\sqrt{[\gamma\gamma]} = \pm\ 7.670842$ $\qquad \sqrt{[\zeta\zeta]} = \pm\ 17478.006$

The solution of the normal equations gives:

$\Delta X = +1.227$	$\Delta\alpha = +0.00029129$	$\Delta r = -0.00029126$
$\Delta Y = +0.609$	$\Delta\omega = -0.00020777$	$\Delta s = -0.00020777$
$\Delta Z = -0.130$	$\Delta\kappa = +0.00003024$	$\Delta t = +0.00001192$

Final solution

$X_o = 582.79 \quad \pm 1.06 = m\sqrt{[\delta\delta]}$

$Y_o = 657.64 \quad \pm 1.14 = m\sqrt{[\epsilon\epsilon]}$ where $m = \pm 9.9\mu$

$Z_o = 2632.00 \quad \pm 0.17 = m\sqrt{[\zeta\zeta]}$

$r = -.01099196 \quad \pm 0.00046837 = m\sqrt{[\alpha\alpha]} \quad \alpha = -0^\circ\ 37'\ 47.3'' \pm 1'\ 35.6''$

$s = -.00377188 \quad \pm 0.00050521 = m\sqrt{[\beta\beta]} \quad \omega = -0^\circ\ 12'\ 58.0'' \pm 1'\ 44.2''$

$t = +.91897650 \quad \pm 0.00007602 = m\sqrt{[\gamma\gamma]} \quad \kappa = 66^\circ\ 46'\ 36.9'' \pm 0'\ 15.7''$

m = mean error of an observation of unit weight after the adjustment.

Computation of residuals:

v	v'	Final Check $G-(\ell+v)$	$G'-(\ell'+v')$
microns	microns	8th decimal place	
-0.33	+1.05	-7	-28
+6.70	+5.44	-6	-13
-13.38	+1.53	-13	-9
+12.77	-0.75	+5	-40
-0.43	+19.18	-1	-23
-2.81	-9.72	0	-32
-1.19	-11.87	+1	-29
-7.82	+1.60	+10	-25
+6.44	-6.26	0	-20

[vv] = 1178.

[pLL.6] = 1177.

Beginning of a third iteration:

$c = 0.16999$

$X_o^o = 582.79$ $\quad r^o = -0.01099196$ $\quad r'^o = -0.99993948$

$Y_o^o = 657.64$ $\quad s^o = -0.00377188$ $\quad s'^o = +0.99999289$

$Z_o^o = +2632.00$ $\quad t^o = +0.91897650$ $\quad t'^o = +0.39431230$

$A^o = +0.39432654$ $\quad A^{o'} = +0.91890454$ $\quad D^o = -0.01099188$

$B^o = -0.91896997$ $\quad B^{o'} = +0.39430950$ $\quad E^o = -0.00377188$

$C^o = -0.00663530$ $\quad C^{o'} = -0.01158856$ $\quad F^o = -0.99993227$

	$X-X_o^o$	$Y-Y_o^o$	$Z-Z_o^o$	m	n	q
1	+635.66	-366.84	-2185.9	+589.6703506	+464.7937962	+2180.15
2	+168.16	-513.84	-2196.1	+556.9615938	-29.7369404	+2196.11
3	-309.59	-657.64	-2191.9	+484.1748651	-518.3963914	+2197.64
4	+511.46	+80.76	-2183.6	+129.3620389	+527.1321309	+2177.53
5	+38.36	-35.04	-2194.4	+49.2322519	+46.8625093	+2193.96
6	-462.24	-265.54	-2189.6	+63.6507967	-504.0850682	+2195.53
7	+347.16	+519.66	-2162.0	-338.7804846	+548.9682416	+2156.08
8	-136.74	+402.76	-2182.3	-422.1498833	+58.4508019	+2182.14
9	-582.79	+377.16	-2193.0	-482.6073185	-400.8278438	+2198.21

	ℓ^{o}	ℓ'^{o}	$\ell^{o} - \ell$	$\ell'^{o} - \ell'$
	$\frac{cm}{g}$	$\frac{cm}{g}$	$= \Delta\ell$	$- \Delta\ell'$
1	+.04597760	+.03624076	-.00000040	+.00000076
2	+.04311164	-.00230179	+.00000664	+.00000521
3	+.03745149	+.04009856	-.00001351	+.00000144
4	+.01009871	+.04115084	+.00001271	-.00000116
5	+.00381456	+.00363095	-.00000044	+.00001895
6	+.00492819	-.03902904	-.00000281	-.00001004
7	-.02671018	+.04328184	-.00000118	-.00001216
8	-.03288573	+.00455335	-.00000773	+.00000135
9	-.03732056	-.03099646	+.00000644	-.00000646

$[\Delta\ell]$ -.00000028 $[\Delta\ell']$ -.00000211

discrimination factor = 0.00000013

If the preestablished discrimination constant was ≤ 0.00000015 the iteration will be stopped at this stage.

SUMMARY

A comparison of the effort involved in the different numerical solutions for the orientation of a photogrammetric camera shows that the solution based on the general expressions explicit in the terms of the observed plate coordinates (formulas (48), pages 37-44 and numerical examples Nos. 2 and 3) is the most economical solution and suited for electronic computing devices. This statement is based on the experience gained from a large number of orientations computed by the different methods. It may be seen directly by comparing the characteristics of this solution with those of other possible approaches as outlined in the present report as well as BRL Report No. 784.[15]

(1) The formulas do not restrict the problem to the projective relation of two planes, but relate the image plane (the photograph) to the object space.

(2) The solution has variables with physical meanings derived from instrument readings (dials and levels) or independent camera calibration procedures.

(3) Consequently, there are no additional transformations necessary from plate constants to orientation elements and vice versa.

(4) Furthermore, the solution may be reduced to the actual number of unknowns, thus reducing the numerical work to a minimum.

(5) No additional condition equations need be introduced.

(6) No additional weights are necessary.

(7) The residuals v and v' are directly obtained from the observational equations.

(8) The mean errors of the orientation elements are directly obtained from the general solution of the normal equations.

(9) The solution is suited for electronic computing devices.

[15] Compare (11)

APPENDIX

After coding the orientation problem for a single photogrammetric camera based on formulas (64) - (75), the corresponding computations have been successfully carried out on the ORDVAC computer at the Ballistic Research Laboratories. A single iteration carrying six unknowns for $n = 10$ points takes 35 seconds. In general, it will be possible to obtain approximate values for the unknowns which will make the solution converge in five iterations or less. Printing out the result on IBM-cards and tabulating about 75 answers (20 residuals v and v', 20 $\Delta \ell$ and $\Delta \ell'$ values for the final check, 6 unknowns and their mean errors, 21 weight coefficients, $[vv]$, $[LL.6]$ and the mean error of an observation of unit weight) takes another 35 seconds. Thus the required time for the analytical solution of the orientation problem for either a spatial resection or a camera calibration takes about 3 minutes or even less. On the following page a copy is given of the original print of the result of example No. 3 as obtained from the ORDVAC after 2 iterations. The result shows complete agreement with the corresponding hand computed values on page 83.

The IBM-output is in the floating decimal point system. The last two figures determine the position of the decimal point with respect to the number given in the first seven figures. The number of the last two digits indicates how many places the decimal point must be moved. No sign in front of these last two figures indicates that the decimal point must be moved to the right, a negative sign indicates a corresponding displacement to the left, e.g.

$$.1004976\text{-}01 = +.01004976$$

$$-.1364183\ 06 = -136418.3$$

IBM - Output

.1004976-01	.4729380-03	.1570972-06			discrimination factors times 2n
-.3207300-06	.6707158-05	-.1338682-04	.1275217-04	-.4318467-06	$v_1 - v_5$
-.2804539-05	-.1216732-05	-.7791497-05	.6448495-05		$v_6 - v_9$
.1048948-05	.5435164-05	.1537130-05	-.8168523-06	.1917667-04	$v_1' - v_5'$
-.9765865-05	-.1185548-04	.1591544-05	-.5220567-05		$v_6' - v_9'$
.0000000-10	.1164153-10	.0000000-10	.0000000-10	.0000000-10	$G - (\ell + v)$
.0000000-10	.0000000-10	.0000000-10	.1164153-10		
.1164153-10	.0000000-10	.1164153-10	.1164153-10	.1164153-10	$G' - (\ell' + v')$
.0000000-10	.0000000-10	.0000000-10	.0000000-10		
-.1099207-01	-.9999395 00	-.3770936-02	.9999928 00	.9189764 00	r, r', s, s', t
.3943123 00	.5827874 03	.5576368 03	.2631998 04		t', X_o, Y_o, Z_o
-.2232913 04	-.2594970 04	-.5883905 02	-.1144703 11	-.1324312 11	$[\alpha\alpha], [\beta\beta], [\gamma\gamma], [\delta\delta], [\epsilon\epsilon]$
-.3054418 09	.1814542 03	-.1843118 02	-.5052602 07	-.4129411 06	$[\xi\xi], [\alpha\beta], [\alpha\gamma], [\alpha\delta], [\alpha\epsilon]$
-.2462817 05	.5512347 02	.4172214 06	.5858061 07	-.2812779 06	$[\alpha\xi], [\beta\gamma], [\beta\delta], [\beta\epsilon], [\beta\xi]$
-.4499332 05	-.1364183 06	.6226412 04	-.9494156 09	-.5111803 03	$[\gamma\delta], [\gamma\epsilon], [\gamma\xi], [\delta\epsilon], [\delta\xi]$
.6275622 09	.1177586-08	.1177617-08	.9906298-05	.4681094-03	$[\epsilon\xi], [LL.6], [vv+v'v'], m, m_r$
.5046352-03	.7598722-04	.1059832 01	.1140004 01	.1731313 00	$m_s, m_t, m_{X_o}, m_{Y_o}, m_{Z_o}$

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